

Group-theoretic Algorithms for Multiphoton Interferometry

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Introduction and Background

Wigner \mathcal{D} -functions are the matrix elements of the representations of finite group $SU(n)$ of $n \times n$ special unitary matrices. Wigner \mathcal{D} -functions of $SU(2)$ group elements are used in nuclear, atomic and molecular physics [1–3]. In the Standard Model of particle physics, $SU(2)$, $SU(3)$ and $SU(6)$ \mathcal{D} -functions are used to describe transformations that preserve global or local symmetries [4–6].

Recently, $SU(n)$ transformations have been the subject of considerable interest because of the BOSONSAMPLING problem. The output from an n -channel passive optical interferometer affecting a $SU(n)$ transformation on indistinguishable single-photon pulse inputs is computationally hard classically subject to conjectures [7–9]. The action of three-channel optical interferometers on partially distinguishable single-photon inputs is best described by $SU(3)$ \mathcal{D} -functions [10–12].

\mathcal{D} -functions of $SU(2)$ group elements are well studied and tabulated [13]. $SU(3)$ \mathcal{D} -functions in a weight basis can be calculated as products of $SU(2)$ Wigner \mathcal{D} -functions [14, 15]. \mathcal{D} -functions in the weight basis, which connect eigenstates of the weight-basis elements of $\mathfrak{su}(n), \mathfrak{su}(n-1) \dots \mathfrak{su}(2)$ Lie algebras, are especially suitable for studying permutational symmetric Bosonic systems. \mathcal{D} -functions in the Gelfand-Tsetlin basis for $\mathfrak{su}(n)$ algebras can be constructed [16, 17] but these functions lack the manifest permutational symmetries that arise in physical systems like BOSONSAMPLING interferometers.

Problem Statement

Despite the importance of \mathcal{D} -function in physics, there is no known procedure to analytically compute the $SU(n)$ Wigner \mathcal{D} -functions for $n \geq 4$ in the weight basis. The unavailability of expressions for the $SU(n)$ Wigner \mathcal{D} -functions hinders us from exploring and exploiting the rich group-theoretic structure of optical interferometry.

In this paper, we (1) devise a symbolic algorithm to compute expressions for states of $SU(n)$ irreducible representations (irreps) in the weight-basis, (2) devise a symbolic algorithm to compute expressions for Wigner \mathcal{D} -functions of $SU(n)$, and (3) employ \mathcal{D} -functions to compute outputs from $SU(4)$ interferometry and relate these outputs to determinants, immanants and permanents of the $SU(n)$ transformation matrices.

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High-level Overview of Algorithms

Our procedure to calculate the expressions for the Wigner \mathcal{D} -function in the weight basis consists of two key algorithms. The first algorithm returns an expression for the weight-basis states corresponding to each weight vector that occurs in the given $SU(n)$ irrep. The second algorithm uses the basis sets constructed in the first algorithm to find the Wigner \mathcal{D} -functions that transform one weight-basis state into another.

We calculate the basis set at each weight state using a graph-theoretic algorithm on the weight graph (multiplet) of the given $SU(n)$ irrep. We define the weight graph $G = (W, E)$ of an $SU(n)$ irrep as follows. The set $W = \{w_i\}$ of vertices is the set of weights of a given irrep of $SU(n)$ and the edges $e_j \in E$ represent the action of the elements of the Lie algebra $\mathfrak{su}(n)$ on the weight-basis states, i.e., $e_j = (w_k, w_\ell) \in E$ iff \exists operator $c \in \mathfrak{su}(n)$ weight basis such that $c(\psi_{w_k}) = \psi_{w_\ell}$, where ψ_{w_k} and ψ_{w_ℓ} are $SU(n)$ basis states with weights w_k and w_ℓ respectively. Each vertex w_i is occupied by a vector space of dimension equal to the multiplicity $M(w_i)$ of the weight [18]. The output from our algorithm is a set of basis state for the vector space at each vertex.

The basis-states construction algorithm starts with a symbolic expression for the highest weight state, which is annihilated by all $\mathfrak{su}(n)$ raising operators and is known to have a multiplicity $M(v_{i_{\text{hws}}}) = 1$. The algorithm then searches the weight graph for $SU(n)$ states using lowering operators of the $\mathfrak{su}(n)$ algebra in an order inspired by breadth-first graph search. If the current vertex in the graph search has multiplicity greater than one, we obtain different states on approaching the vertex along different paths and employ the Gram-Schmidt procedure to obtain an orthonormal basis set. The algorithm halts when a basis set at each vertex is found and returns a list of expressions for the basis states of the vector space at each vertex.

The second algorithm calculates the Wigner \mathcal{D} -functions that transform a basis state of one given $SU(n)$ irrep label to a basis state with another given irrep label. We compute the expressions for basis states corresponding to the two given labels. Next, we compute the expression for the transformed (by the given $SU(n)$ matrix element) basis state by pattern matching [19]. The overlap between the second basis state and the transformed first basis state is the Wigner \mathcal{D} -function of interest. The \mathcal{D} -functions obtained by the algorithm were successfully verified by numerically comparing with approximate \mathcal{D} -functions computed by exponentiating elements of the algebra.

Application to Four-Channel Passive Optical Interferometer

We use $SU(4)$ Wigner \mathcal{D} -functions that were calculated by our algorithm to determine the action of a four-channel passive optical interferometer when controllably-delayed single-photon pulse are incident at each input port. We determine four-photon coincidence rates around zero time delay in terms of $SU(4)$ Wigner \mathcal{D} -functions. We connect temporal distinguishability of input photons with linear combinations of immanants

of the $SU(4)$ interferometer matrix. Immanants of a matrix are quantities of varying computational complexity [20] and generalize the permanent and determinant, which are relevant for input states with permutation symmetry. Using Wigner \mathcal{D} -functions leads to a reduction in the computation cost of coincidence landscapes and the connection with immanants allows for a rigorous computational complexity analysis of the optical interferometer.

Conclusion and Potential Impact

In summary, we have devised symbolic algorithms to compute expressions for states of $SU(n)$ irreps and expressions for $SU(n)$ Wigner \mathcal{D} -functions in the weight-basis. We have used the obtained \mathcal{D} -functions to compute coincidence landscapes of $SU(4)$ interferometry and relate these landscapes to determinants, immanants and permanents of the $SU(4)$ transformation matrix.

Our \mathcal{D} -function calculation algorithm opens the possibility of applying graph-theoretic methods to $SU(n)$ group theory. This work also generalizes passive optical interferometry beyond the three-photon level [10–12]. We find the connection between coincidence landscapes and immanants of $SU(n)$ transformation matrices, thereby generalizing the matrix permanent analysis of the `BOSONSAMPLING` problem.

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