Group-theoretic Algorithms for Multiphoton Interferometery

Ish Dhand*, Hubert de Guise†, Barry C. Sanders*‡§

October 25, 2014

Introduction and Background

Wigner $\mathcal{D}$-functions are the matrix elements of the representations of finite group SU(n) of $n \times n$ special unitary matrices. Wigner $\mathcal{D}$-functions of SU(2) group elements are used in nuclear, atomic and molecular physics [1–3]. In the Standard Model of particle physics, SU(2), SU(3) and SU(6) $\mathcal{D}$-functions are used to describe transformations that preserve global or local symmetries [4–6].

Recently, SU(n) transformations have been the subject of considerable interest because of the BosonSampling problem. The output from an n-channel passive optical interferometer affecting a SU(n) transformation on indistinguishable single-photon pulse inputs is computationally hard classically subject to conjectures [7–9]. The action of three-channel optical interferometers on partially distinguishable single-photon inputs is best described by SU(3) $\mathcal{D}$-functions [10–12].

$\mathcal{D}$-functions of SU(2) group elements are well studied and tabulated [13]. SU(3) $\mathcal{D}$-functions in a weight basis can be calculated as products of SU(2) Wigner $\mathcal{D}$-functions [14, 15]. $\mathcal{D}$-functions in the weight basis, which connect eigenstates of the weight-basis elements of $\mathfrak{su}(n), \mathfrak{su}(n-1) \ldots \mathfrak{su}(2)$ Lie algebras, are especially suitable for studying permutational symmetric Bosonic systems. $\mathcal{D}$-functions in the Gelfand-Tsetlin basis for $\mathfrak{su}(n)$ algebras can be constructed [16, 17] but these functions lack the manifest permutational symmetries that arise in physical systems like BosonSampling interferometers.

Problem Statement

Despite the importance of $\mathcal{D}$-function in physics, there is no known procedure to analytically compute the SU(n) Wigner $\mathcal{D}$-functions for $n \geq 4$ in the weight basis. The unavailability of expressions for the SU(n) Wigner $\mathcal{D}$-functions hinders us from exploring and exploiting the rich group-theoretic structure of optical interferometry.

In this paper, we (1) devise a symbolic algorithm to compute expressions for states of SU(n) irreducible representations (irreps) in the weight-basis, (2) devise a symbolic algorithm to compute expressions for Wigner $\mathcal{D}$-functions of SU(n), and (3) employ $\mathcal{D}$-functions to compute outputs from SU(4) interferometery and relate these outputs to determinants, immanants and permanents of the SU(n) transformation matrices.

*IQST, University of Calgary
†Lakehead University
‡University of Science and Technology China
§CIFAR

High-level Overview of Algorithms

Our procedure to calculate the expressions for the Wigner $\mathcal{D}$-function in the weight basis consists of two key algorithms. The first algorithm returns an expression for the weight-basis states corresponding to each weight vector that occurs in the given SU(n) irrep. The second algorithm uses the basis sets constructed in the first algorithm to find the Wigner $\mathcal{D}$-functions that transform one weight-basis state into another.

We calculate the basis set at each weight state using a graph-theoretic algorithm on the weight graph (multiplet) of the given SU(n) irrep. We define the weight graph $G = (W, E)$ of an SU(n) irrep as follows. The set $W = \{w_i\}$ of vertices is the set of weights of a given irrep of SU(n) and the edges $e_j \in E$ represent the action of the elements of the Lie algebra $\mathfrak{su}(n)$ on the weight-basis states, i.e., $e_j = (w_k, w_\ell) \in E$ iff $\exists$ operator $c \in \mathfrak{su}(n)$ weight basis such that $c(\psi_{w_k}) = \psi_{w_\ell}$, where ψ_{w_k} and ψ_{w_ℓ} are SU(n) basis states with weights w_k and w_ℓ respectively. Each vertex w_i is occupied by a vector space of dimension equal to the multiplicity $M(w_i)$ of the weight [18]. The output from our algorithm is a set of basis state for the vector space at each vertex.

The basis-states construction algorithm starts with a symbolic expression for the highest weight state, which is annihilated by all $\mathfrak{su}(n)$ raising operators and is known to have a multiplicity $M(v_{i_{\text{hws}}}) = 1$. The algorithm then searches the weight graph for SU(n) states using lowering operators of the $\mathfrak{su}(n)$ algebra in an order inspired by breadth-first graph search. If the current vertex in the graph search has multiplicity greater than one, we obtain different states on approaching the vertex along different paths and employ the Gram-Schmidt procedure to obtain an orthonormal basis set. The algorithm halts when a basis set at each vertex is found and returns a list of expressions for the basis states of the vector space at each vertex.

The second algorithm calculates the Wigner $\mathcal{D}$-functions that transform a basis state of one given SU(n) irrep label to a basis state with another given irrep label. We compute the expressions for basis states corresponding to the two given labels. Next, we compute the expression for the transformed (by the given SU(n) matrix element) basis state by pattern matching [19]. The overlap between the second basis state and the transformed first basis state is the Wigner $\mathcal{D}$-function of interest. The $\mathcal{D}$-functions obtained by the algorithm were successfully verified by numerically comparing with approximate $\mathcal{D}$-functions computed by exponentiating elements of the algebra.

Application to Four-Channel Passive Optical Interferometer

We use SU(4) Wigner $\mathcal{D}$-functions that were calculated by our algorithm to determine the action of a four-channel passive optical interferometer when controllably-delayed single-photon pulse are incident at each input port. We determine four-photon coincidence rates around zero time delay in terms of SU(4) Wigner $\mathcal{D}$-functions. We connect temporal distinguishability of input photons with linear combinations of immanants of the SU(4) interferometer matrix. Immanants of a matrix are quantities of varying computational complexity [20] and generalize the permanent and determinant, which are relevant for input states with permutation symmetry. Using Wigner $\mathcal{D}$-functions leads to a reduction in the computation cost of coincidence landscapes and the connection with immanants allows for a rigorous computational complexity analysis of the optical interferometer.

Conclusion and Potential Impact

In summary, we have devised symbolic algorithms to compute expressions for states of SU(n) irreps and expressions for SU(n) Wigner $\mathcal{D}$-functions in the weight-basis. We have used the obtained $\mathcal{D}$-functions to compute coincidence landscapes of SU(4) interferometry and relate these landscapes to determinants, immanants and permanents of the SU(4) transformation matrix.

Our $\mathcal{D}$-function calculation algorithm opens the possibility of applying graph-theoretic methods to SU(n) group theory. This work also generalizes passive optical interferometry beyond the three-photon level [10–12]. We find the connection between coincidence landscapes and immanants of SU(n) transformation matrices, thereby generalizing the matrix permanent analysis of the BosonSampling problem.

References

[1] G. Racah. Theory of Complex Spectra. II. *Phys. Rev.*, 62(9-10):438, 1942.

[2] M. Jacob and G. C. Wick. On the General Theory of Collisions for Particles with Spin. *Ann. Phys. (N. Y).*, 7(4):404–428, 1959.

[3] K. Alder, A. Bohr, T. Huus, B. Mottelson, and A. Winther. Study of Nuclear Structure by Electromagnetic Excitation with Accelerated Ions. *Rev. Mod. Phys.*, 1956.

[4] S. I. So and D. Strottman. Wigner Coefficients for SU (6) $\supseteq$ SU(3) $\otimes$ SU(2). *J. Math. Phys.*, 20(1):153–176, 1979.

[5] K. Hecht and W. Zahn. An SU(3) Approach to Nuclear Multi-Cluster Problems. *Nucl. Phys. A*, 1979.

[6] K. T. Hecht and Y. Suzuki. Some Special SU(3) $\supseteq$ R(3) Wigner coefficients and their Application. *J. Math. Phys.*, 24(4), 1983.

[7] S. Aaronson and A. Arkhipov. BosonSampling Is Far From Uniform. 2013, 1309.7460.

[8] S. Aaronson and A. Arkhipov. The Computational Complexity of Linear Optics. *Proc. 43rd Annu. ACM Symp. Theory Comput. - STOC '11*, page 333, 2011.

[9] S. Aaronson and A. Arkhipov. The Computational Complexity of Linear Optics. 2010, 1011.3245.

[10] S.-H. Tan, Y. Y. Gao, H. de Guise, and B. C. Sanders. SU(3) Quantum Interferometry with Single-Photon Input Pulses. *Phys. Rev. Lett.*, 2013.

[11] H. de Guise, S.-H. Tan, I. P. Poulin, and B. C. Sanders. Coincidence Landscapes for Three-Channel Linear Optical Networks. *Phys. Rev. A*, 89(6):63819, 2014.

[12] M. Tillmann, S.-H. Tan, S. E. Stoeckl, B. C. Sanders, H. de Guise, R. Heilmann, S. Nolte, A. Szameit, and P. Walther. BosonSampling with Controllable Distinguishability. 2014, 1403.3433.

[13] D. A. Varšalovič, A. N. Moskalev, and V. K. Kersonskij. *Quantum Theory of Angular Momentum*. World Scientific, 1989.

[14] E. Chacón and M. Moshinsky. Comments on Charge Distribution in Multipion Production. *Phys. Rev. D*, 1973.

[15] D. J. Rowe, B. C. Sanders, and H. de Guise. Representations of the Weyl group and Wigner functions for SU(3). *J. Math. Phys.*, 40(7):3604, 1999.

[16] A. Molev. A Basis for Representations of Symplectic Lie algebras. *Commun. Math. Phys.*, 618(1999):591–618, 1999.

[17] A. Molev. Weight bases of Gelfand-Tsetlin type for Representations of Classical Lie algebras. *J. Phys. A. Math. Gen.*, 4143, 2000.

[18] B. Kostant. A Formula For the Multiplicity of a Weight. *Trans. Am. Math. Soc.*, pages 53–73, 1959.

[19] D. E. Knuth, J. J. H. Morris, and V. R. Pratt. Fast Pattern Matching in Strings. *SIAM J. Comput.*, 6(2):323–350, 1977.

[20] P. Bürgisser. The Computational Complexity of Immanants. *SIAM J. Comput.*, 30(3):1023–1040, 2000.