No-signalling assisted zero-error communication via quantum channels and the Lovász Θ number

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If you've been partying...
Hungover Summary

1. $C_0(G) \leq \log \nu(G)$

2. $C_{OE}(G) \leq \log \nu(G)$

3.-5. $C_{ONS}(G) = \log \nu(G)$
Hungover Summary

1. $C_0(G) \leq \log \psi(G)$

2. $C_{OE}(G) \leq \log \psi(G)$

3. $C_{ONS}(G) = \log \psi(G)$

Zero-error capacity of the graph $G$

Lovász number; it's a semidefinite programme
Hungover Summary

1. $C_0(G) \leq \log \delta(G)$
2. $C_{OE}(G) \leq \log \delta(G)$
3. $C_{NS}(G) = \log \delta(G)$

Zero-error capacity
of the graph $G$

Lovász number;
it's a semidefinite
programme

Can be <
Might be <

Yes, it's equality!
1. Channels & graphs

Channel $N : X \rightarrow Y$, i.e. stochastic map

$N(y|x) : \text{transition probabilities}$
1. Channels & graphs

Channel $N : X \rightarrow Y$, i.e. stochastic map

$X \ni x \xrightarrow{N} y \in Y$

$N(y|x) :$ transition probabilities

Want to send information (in $x$), such that receiver (seeing $y$) can be certain about it.
1) Transition graph $\Gamma$: bipartite graph on $X \times Y$ with adjacency matrix

\[
\Gamma(y|x) = \begin{cases} 
1 & \text{if } N(y|x) > 0, \\
0 & \text{if } N(y|x) = 0.
\end{cases}
\]
1) Transition graph \( \Gamma \): bipartite graph on \( X \times Y \) with adjacency matrix

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2) Confusability graph \( G \) on \( X \): adj. matrix

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Lovász convention:

$x \sim x'$ iff $x = x'$ or $xx'$ edge
Example?
\[ \Gamma = T_5 \]

typewriter channel

\[ G = C_5 \]

pentagon
$\Gamma = T_3$

$G = K_3$
$\Gamma = T_3$

$\Gamma = \ast$

$G = K_3$
Product channels:
\[ N_x N'(y y'|x x') = N(y|x) N'(y'|x') \]
Product channels:

\[ N \times N'(yy'|xx') = N(y|x)N'(y'|x') \]

Graphs via Kronecker/tensor product:

\[ \Gamma(N \times N') = \Gamma \otimes \Gamma' \]

\[ 1 + A(N \times N') = (1 + A) \otimes (1 + A') \]
Product channels:
\[ N \times N' (y y' | x x') = N(y | x) N'(y' | x') \]

Graphs via Kronecker/tensor product:
\[ \Gamma(N \times N') = \Gamma \otimes \Gamma' \]
\[ 1 + A(N \times N') = (1 + A) \otimes (1 + A') \]

Strong graph product \( G \times G' \)
Zero-error transmission

\[ \frac{1}{2} \]

\[ i \rightarrow x = f(i) \rightarrow N \rightarrow y \text{ possible: } N(y|x) > 0 \]
1/2. Zero-error transmission

\[ i \xrightarrow{x = \mathcal{A}(i)} N \xrightarrow{\text{y possible:}} N(y|x) > 0 \]

Hence: codebook \( \mathcal{A}(i) \subseteq X \) must be an independent set in \( G \).

Maximum size:

\[ \alpha(G) := \text{independence number of } G. \]
1\frac{1}{2}. Zero-error transmission

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Hence: codebook \( \{f(i)\}\) \( \subset X \) must be an independent set in \( G \).

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Well-known to be \( \text{NP-complete} \)!
Zero-error transmission

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Maximum size:

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Well-known to be NP-complete!

Upper bounds!?
\[ \alpha(G) \leq \vartheta(G) = \max \text{ Tr } BJ \text{ s.t. } B \geq 0, \text{ Tr } B = 1 , \\
B_{xy} = 0 \ \forall xy \in G. \]

\[ \alpha(G) \leq \vartheta(G) = \max \text{ Tr } B \mathcal{J} \text{ s.t. } B \succeq 0, \text{ Tr } B = 1, \]
\[ B_{xy} = 0 \ \forall xy \in G. \]

[\text{L. Lovász, IEEE-IT 25(1):1-7, 1979}]

\[ \leq \alpha^*(\Gamma) = \max \sum_x \omega_x \ \text{s.t. } \omega_x \geq 0 \ \& \ \\
\forall y \sum_x \Gamma(y|x)\omega_x \leq 1. \]

[\text{C.E. Shannon, IRE-IT 2(3):8-19, 1956}]

\[ \alpha(G) \leq \vartheta(G) = \max \ Tr \ BJ \ \text{s.t.} \ B \geq 0, \ Tr \ B = 1, \ B_{xy} = 0 \ \forall xy \in G. \]


\[ \leq \alpha^*(\Gamma) = \max \sum_{x} \omega_{x} \ \text{s.t.} \ \omega_{x} \geq 0 \ \& \ \forall y \sum_{x} \Gamma(y|x)\omega_{x} \leq 1. \]

[C.E. Shannon, IRE-IT 2(3):8-19, 1956]

Best: \[ \alpha^*(G) = \min \ \alpha^*(\Gamma) \ \text{s.t.} \ G \supset \text{graph of } \Gamma \]
$$\alpha(G) \leq \chi(G) = \max \operatorname{Tr} B J \text{ s.t. } B \geq 0, \operatorname{Tr} B = 1, B_{xy} = 0 \ \forall xy \in G.$$


$$\leq \alpha^*(\Gamma) = \max \sum_x \omega_x \text{ s.t. } \omega_x \geq 0 \ \& \ \forall y \sum_x \Gamma(y|x)\omega_x \leq 1.$$  

[C.E. Shannon, IRE-IT 2(3):8-19, 1956]

Best: $$\alpha^*(G) = \min \alpha^*(\Gamma) \text{ s.t. } G \supset \text{graph of } \Gamma$$

(Attained at $\Gamma$ that has an output for every maximal clique of $G$: $\Gamma(C|x) = 1$ iff $x \in C$.)
Asymptotically many channel uses - capacity:

\[ C(G) = \lim_{n \to \infty} \frac{1}{n} \log \alpha(G^x^n) \]

[C.E. Shannon, IRE-IT 2(3):8-19, 1956]
Asymptotically many channel uses - capacity:

\[ C_0(G) = \lim_{n \to \infty} \frac{1}{n} \log \alpha(G^{\times n}) \]

\[ = \sup \text{ because } \alpha(G \times H) \geq \alpha(G) \alpha(H) \]

\[ \text{[C.E. Shannon, IRE-IT x(3):8-19, 1956]} \]
Asymptotically many channel uses - capacity:

\[ C(G) = \lim_{n \to \infty} \frac{1}{n} \log \alpha(G^{\times n}) \leq \log \chi(G) \]

\[ \chi(G \times H) = \chi(G) \chi(H) ! \]

\[ \chi(G) = \sup \{ \chi(G^G) \} \]

\[ \chi(G) \geq \chi(G) \chi(H) \]

\[ \text{C.E. Shannon, IRE-IT } 2(3):8-19, 1956 \]

\[ \text{L. Lovász, IEEE-IT } 25(1):1-7, 1979 \]
\[
\log \alpha(G) \leq C_0(G) \leq \log \kappa(G) \leq \log \alpha^*(\Gamma)
\]

Also fractional packing number multiplicative:

\[
\alpha^*(\Gamma \times \Gamma') = \alpha^*(\Gamma) \alpha^*(\Gamma'),
\]

\[
\alpha^*(G \times H) = \alpha^*(G) \alpha^*(H)
\]
\[ \log \alpha(G) \leq \mathcal{C}_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma) \]

All inequalities can be strict; first and last:

Ex. Typewriter channel/pentagon

\[ \alpha(C_5) = 2, \quad \alpha(C_5 \times C_5) = 5 > 4, \]

but \[ \vartheta(C_5) = \sqrt{5}, \] and \[ \alpha^*(T_5) = 5/2. \]
\[ \log \alpha(G) \leq C_0(G) \leq \log \Theta(G) \leq \log \alpha^*(\Gamma) \]

All inequalities can be strict; first and last:

Ex. Typewriter channel/pentagon
\[ \alpha(C_5) = 2, \quad \alpha(C_5 \times C_5) = 5 > 4, \]
but \( \Theta(C_5) = \sqrt{5} \), and \( \alpha^*(T_5) = 5/2 \).

Note: \( \alpha^*(T_3) = \frac{3}{2} \), but \( \alpha^*(\ast) = 1! \)
\[
\log \alpha(G) \leq c_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma)
\]

All inequalities can be strict; first and last:

Random graphs \( G \sim G(n, \frac{1}{2}) \) have, whp,
\[
\alpha(G) \approx \log n, \quad \vartheta(G) \approx \sqrt{n}, \quad \alpha^*(G) \approx n/(\log n)
\]
\[ \log \alpha(G) \leq C_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma) \]

All inequalities can be strict; middle due to W. Haemers [IEEE-IT 25(2);231-232, 1979], via a different algebraic and multiplicative bound on \( \alpha \) which sometimes(!) is better than \( \vartheta \).
log α(G) ≤ C₀(G) ≤ log δ(G) ≤ log α*(Γ)

All inequalities can be strict; middle due to W. Haemers [IEEE-IT 25(2);231-232, 1979], via a different algebraic and multiplicative bound on δ which sometimes(!) is better than δ.

However: w/o sacrificing multiplicativity, δ cannot be improved [Acín/Duan/Sainz/AW, 2014].
\[ \log \alpha(G) \leq C_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma) \]

All inequalities can be strict; middle due to W. Haemers [IEEE-IT 25(2):231-232, 1979], via a different algebraic and multiplicative bound on \( \vartheta \) which sometimes(!) is better than \( \vartheta \).

Determination of \( C_0(G) \) open, not even known to be computable...

[N. Alon/E. Lubetzky, IEEE-IT 52(5):2172-2176, 2006]
$$C_0(G) \leq \log \mathcal{V}(G)$$

Idea: Perhaps we can close the gap by allowing additional resources in the en-/ decoding?
$$C_0(G) \leq \log \vartheta(G)$$

Idea: Perhaps we can close the gap by allowing additional resources in the en-/decoding?

+ feedback [C.E. Shannon, IRE-IT 2(3):8-19, 1956]

\[ C_{or}(\Gamma) = \log \alpha^*(\Gamma), \text{ with constant activating noiseless bits.} \]
\[ C_0(G) \leq \log \chi(G) \]

Idea: Perhaps we can close the gap by allowing additional resources in the encoding/decoding?

+ feedback [C.E. Shannon, IRE-IT 2(3):8-19, 1956]
+ entanglement (quantum correlations)
+ no-signalling correlations
2. Free non-local resources
For instance, with free entanglement:

\[
\begin{array}{ccc}
A^i_x & \psi & B^y_j \\
\downarrow & \psi & \uparrow \\
x & \downarrow & \uparrow y \\
\end{array}
\]

Maximum #messages $=: \tilde{\alpha}(G)$

Can show that this depends only on $G$; furthermore can be $> \alpha(G)$...

[T.S. Cubitt et al., PRL 104:230503, 2010]
Known: $\alpha(G) \leq \bar{\alpha}(G) \leq \chi(G)$

Known: $\alpha(G) \leq \tilde{\alpha}(G) \leq \imath(G)$

Since $\imath$ is multiplicative under strong graph product, $\imath(G \times H) = \imath(G) \cdot \imath(H)$, get:

$$C_o(G) \leq C_{oE}(G) = \lim_{n \to \infty} \frac{1}{n} \log \tilde{\alpha}(G^\times n) \leq \log \imath(G)$$
Known: $\alpha(G) \leq \tilde{\alpha}(G) \leq \chi(G)$

Since $\chi$ is multiplicative under strong graph product, $\chi(G \times H) = \chi(G) \chi(H)$, get:

$$C_0(G) \leq C_{OE}(G) = \lim_{n \to \infty} \frac{1}{n} \log \tilde{\alpha}(G^\times n) \leq \log \chi(G)$$

Known examples of separation

Known: \( \alpha(G) \leq \tilde{\alpha}(G) \leq \nu(G) \)

Since \( \nu \) is multiplicative under strong graph product, \( \nu(G \times H) = \nu(G) \nu(H) \), get:

\[
C_0(G) \leq C_{0E}(G) = \lim \frac{1}{n} \log \tilde{\alpha}(G^n) \leq \log \nu(G)
\]

Unknown whether = or < !

Known examples of separation
Allowing general no-signalling correlation:

This is no-signalling assisted zero-error code if \( j = i \) with probability 1.

I.e., for all \( j \neq i \) & edges \( xy \) in \( \Gamma \), \( P(x|yi |y) = 0 \)
Allowing general no-signalling correlation:

This is no-signalling assisted zero-error code if \( j = i \) with probability 1.

I.e., for all \( j \neq i \) & edges \( xy \) in \( \Gamma \), \( P(x|j|i|y) = 0 \)

Maximum #msg. with \( P \in NS =: \overline{\alpha}(\Gamma) \)
\( \overline{\alpha}(\Gamma) = \max m \text{ s.t. } \mathcal{P}(x_j|y) \in NS, \ ij=1...m, \forall i \neq j \forall xy \in \Gamma \mathcal{P}(x_j|y)=0. \)
Clear: Can test given $m$ efficiently by linear programming. Less obvious:
\[ \overline{\alpha}(\Gamma) = \max m \text{ s.t. } P(x_j|y_i) \in NS, \ i,j=1,...,m, \ \forall i \neq j \forall y \in \Gamma \ P(x_j|y_i) = 0. \]

Clear: Can test given \( m \) efficiently by linear programming. Less obvious:

Thm. \[ \overline{\alpha}(\Gamma) = \left\lfloor \alpha^*(\Gamma) \right\rfloor, \text{ with } \alpha^* \text{ the fractional packing number of } \Gamma: \]
\[ \alpha^*(\Gamma) = \max \sum_x w_x \text{ s.t. } w_x \geq 0 \text{ & for all } y, \sum_x \Gamma(y|x)w_x \leq 1. \]

[T.S. Cubitt et al., IEEE-IT 57(8):5509-5523, 2011]
Thm. $\overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$, with $\alpha^*$ the fractional packing number of $\Gamma$: $\alpha^*(\Gamma) = \max \sum_{x} \omega_{x}$ s.t. $\omega_{x} \geq 0$ & for all $y$, $\sum_{x} \Gamma(y|x)\omega_{x} \leq 1$.

[T.S. Cubitt et al., IEEE-IT 57(8):5509-5523, 2011]

Cor. Due to multiplicativity of $\alpha^*$, $C_{ONs}(\Gamma) = \lim \frac{1}{n} \log \overline{\alpha}(\Gamma^\otimes n) = \log \alpha^*(\Gamma)$.

[C.E. Shannon, 1956: Same answer for feedback-assisted capacity!]

...so this is too big - what now?!
3. Quantum version...
Should really consider quantum channels $N : B(A) \rightarrow B(B)$, cptp map on states:

$$\rho \xrightarrow{N} \sigma = N(\rho)$$

Kraus form: $N(\rho) = \sum_i E_i \rho E_i^\dagger$, $\sum_i E_i^\dagger E_i = 1$
For quantum channel (cptp map) \( N : B(A) \rightarrow B(B) \), with Kraus op's \( E_i \):

Define \( K = \operatorname{span} \mathbb{E}_i \mathbb{E}_j \subset B(A \rightarrow B) \) and \( S = K^\dagger K = \operatorname{span} \mathbb{E}_i \mathbb{E}_j^\dagger \mathbb{E}_j \mathbb{E}_i \subset B(A) \) as natural analogues of the transition and confusability graphs.

For quantum channel (cptp map) $N : B(A) \to B(B)$, with Kraus op’s $E_i$:

Define $K = \operatorname{span}\{E_i \} \subseteq B(A \to B)$ and $S = K^\dagger K = \operatorname{span}\{E_i \dagger E_j \} \subseteq B(A)$ as natural analogues of the transition and confusability graphs.

($S = S^\dagger \equiv 1$, so $S$ is an operator system)

Define $K = \text{span}\{E_i\}_i \subset \mathcal{B}(A \rightarrow B)$ and $S = K^+ K = \text{span}\{E_i^+ E_j\}_i \subset \mathcal{B}(A)$.

For classical channel, Kraus operators are $\propto \Gamma(y|x) \, |y><x|$, so:

$$K = \text{span}\{\Gamma(y|x) \, |y><x|\}_y \quad \leftrightarrow \quad \Gamma,$$

$$S = \text{span}\{|x'><x|\, \text{s.t. } x \sim x'\}_x \quad \leftrightarrow \quad G.$$

...hence $S, K$ extend $G, \Gamma$ to quantum...

Define $K = \text{span} E_i \otimes \mathbb{I} \subset \mathcal{B}(A \rightarrow B)$ and $S = K^\dagger K = \text{span} E_i^\dagger E_i \otimes \mathbb{I} \subset \mathcal{B}(A)$.

Can show: Zero-error transmission assisted by entanglement (or without) depends only on $S$.

Below treat assistance by quantum no-signalling correlations, which will turn out to depend only on $K$.

Quantum no-signalling:

No-signalling means:

\[ \text{Tr}_U \mathcal{P}(\sigma \otimes \tau) = B(\tau), \]

\[ \text{Tr}_V \mathcal{P}(\sigma \otimes \tau) = A(\sigma). \]

[D. Beckman et al., PRA 64:052309, 2001; T. Eggeling et al., Europhy. Lett. 57(6):782-788, 2002; M. Piani et al., PRA 74:012305, 2006]

Cf. W. Matthews’ talk on Wed!
Quantum no-signalling:

No-signalling means:

\[ \text{Tr}_U P(\sigma \otimes \tau) = B(\tau), \]
\[ \text{Tr}_V P(\sigma \otimes \tau) = A(\sigma). \]

Equiv.: \( P \) linear combination of \( A \otimes B \) plus semidef. constraint for "cptp"

[D. Beckman et al., PRA 64:052309, 2001; T. Eggeling et al., Europhy. Lett. 57(6):782-788, 2002; M. Piani et al., PRA 74:012305, 2006]
Quantum no-signalling:

Although formally a channel with two simultaneous inputs, the no-signalling condition ensures that Alice can use her "box" without waiting. Bob is left with a conditional channel...

Cf. W. Matthews' talk on Wed!
No-signalling assisted communication:

\[ j = i \text{ w.p. 1} \]
No-signalling assisted communication:

Maximum #msg. with $P \in \text{NS} =: \overline{\alpha}(K)$
No-signalling assisted communication:

Maximum #msg. with \( P \in \text{NS} =: \overline{\alpha}(K) \)

Similar definitions of \( \alpha(S) \) and \( \tilde{\alpha}(S) \) via max. # of messages; all reducing to previous notions for classical channels.
No-signalling assisted communication:

\[ i \xrightarrow{P} j \ (= i \text{ w.p. 1}) \]

Thm. [RD/AW]  \( \bar{\alpha}(K) = \left[ \gamma(K) \right] \), where

\[ \gamma(K) = \max \text{ Tr } S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes 1, \]
\[ \text{Tr}_A U = 1^B, \]
\[ \Pi(S \otimes 1 - U) = 0. \]
No-signalling assisted communication:

Thm. [RD/AW] \( \overline{\alpha}(K) = \left[ \gamma(K) \right] \), where

\[
\gamma(K) = \max \text{ Tr } S \text{ s.t. } 0 \leq U_{AB}^{A} \leq S \otimes 1,
\]

\[
\text{Tr}_{A} U = 1_{B}^{B},
\]

\[
\Pi(S \otimes 1 - U) = 0.
\]
\( \gamma(K) = \max \text{Tr} \ S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes 1 \),
\[ \text{Tr}_A U = 1^B, \]
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...reduces to classical fractional packing number for classical channel.

[T.S. Cubitt et al., IEEE-IT 57(8):5509-5523, 2011]
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$$\text{Tr}_A U = 1^B,$$

$$\Pi(S \otimes 1 - U) = 0.$$

...reduces to classical fractional packing number for classical channel.

However, in general much more complex; for instance not multiplicative, i.e.

$$\gamma(K \otimes K') \geq \gamma(K) \gamma(K'),$$

sometimes strict.
\[ \Upsilon(K) = \max \text{Tr } S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes 1, \]
\[ \text{Tr}_A U = 1^B, \]
\[ \Pi(S \otimes 1 - U) = 0. \]

...reduces to classical fractional packing number for classical channel.

However, in general much more complex; for instance not multiplicative, i.e.
\[ \Upsilon(K \otimes K') \geq \Upsilon(K) \Upsilon(K'), \text{ sometimes strict.} \]

What is \( \mathcal{C}_{\text{ONS}}(K) = \lim \frac{1}{n} \log \Upsilon(K^\otimes n) \)?
No-signalling assisted channel simulation:
No-signalling assisted channel simulation:

Any channel with Kraus op's in $K$
No-signalling assisted channel simulation:

Any channel with Kraus op's in \( K \)

**Thm. [RD/AW]** Min #msg = \( \left\lceil \Sigma(K) \right\rceil \), w/ 
\( \Sigma(K) = \min \text{Tr} T \text{s.t.} 0 \leq V^{AB} \leq 1 \otimes T, \) 
\( \text{Tr}_B V = 1^A, \Pi^\perp V = 0. \)
No-signalling assisted channel simulation:

\[ \equiv \begin{array}{c}
\text{Any channel with} \\
\text{Kraus op's in } K
\end{array} \]

**Thm. [RD/AW]** Min #msg = \( \lfloor \Sigma(K) \rfloor \), w/ 
\[ \Sigma(K) = \min \text{ Tr } T \text{ s.t. } 0 \leq V^{AB} \leq 1 \otimes T, \]
\[ \text{Tr}_B V = 1^A, \Pi_v V = 0. \]

...reduces to \( \alpha^*(\Gamma) \) for classical channels.
These are still "very classical", e.g. have confusability graph $G$, $x \sim x'$ iff $\Pi_x \Pi_{x'} \neq 0$. 
4. Cq-channels

\[ |x\rangle\langle x'| \xrightarrow{\mathcal{N}} \delta_{xx'}, \rho_x \]

These are still "very classical", e.g. have confusability graph \( G \), \( x\sim x' \) iff \( \Pi_x \Pi_{x'} \neq 0 \).

\[ \Pi = \sum_x |x\rangle\langle x| \otimes \Pi_x \] Choi matrix support projection, simplifies SDP \( \Upsilon(K) \)...
$$\gamma(K) = \max \quad \text{Tr} \quad S \quad \text{s.t.} \quad 0 \leq U^{AB} \leq S \otimes 1,$$

$$\text{Tr}_A \quad U = 1^B,$$

$$\Pi(S \otimes 1 - U) = 0.$$
\[ \gamma(k) = \max \sum s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x^\perp, \]
\[ \sum (R_x + s_x \Pi_x) = 1. \]
\[ \gamma(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x, \]
\[ \sum_x (R_x + s_x \Pi_x) = 1. \]

\[ \leq A(K) := \max \sum_x s_x \text{ s.t. } 0 \leq s_x, \]
\[ \sum_x s_x \Pi_x \leq 1. \]
\[ \gamma(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_{x}, \]
\[ \sum_x (R_x + s_x \Pi_x) = 1. \]

\[ \leq A(K) : = \max \sum_x s_x \text{ s.t. } 0 \leq s_x, \]
\[ \sum_x s_x \Pi_x \leq 1. \]

Semidefinite packing number; also reduces to fractional packing no. in classical case, but is multiplicative: \( A(K \otimes K') = A(K)A(K') \).
\[ \Upsilon(K) = \max \sum s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x, \]
\[ \sum (R_x + s_x \Pi_x) = 1. \]
\[ \leq A(K) := \max \sum s_x \text{ s.t. } 0 \leq s_x, \]
\[ \sum s_x \Pi_x \leq 1. \]

Thm. \( C_{\text{ONS}}(K) = \lim \frac{1}{n} \log \Upsilon(K^{\otimes n}) = \log A(K). \)
\[
\gamma(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x^\perp, \\
\sum_x (R_x + s_x \Pi_x) = 1.
\]

\[
\leq \mathcal{A}(K) := \max \sum_x s_x \text{ s.t. } 0 \leq s_x, \\
\sum_x s_x \Pi_x \leq 1.
\]

**Thm.** \( \mathcal{C}_{\text{ons}}(K) = \lim_{n} \frac{1}{n} \log \gamma(K^{\otimes n}) = \log \mathcal{A}(K). \)

Show actually \( \gamma(K^{\otimes n}) \geq n^{-c} \mathcal{A}(K)^n \), starting from an optimal solution for \( \mathcal{A}(K); \) then by group (permutation) symmetry that we can satisfy the extra constraints loosing little..
\[
\gamma(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x, \\
\sum_x (R_x + s_x \Pi_x) = 1.
\]

\[
\leq A(K) := \max \sum_x s_x \text{ s.t. } 0 \leq s_x, \\
\sum_x s_x \Pi_x \leq 1.
\]

**Thm.** \( C_{\text{ONS}}(K) = \lim \frac{1}{n} \log \gamma(K^{\otimes n}) = \log A(K). \)

**Thm.** \( G_{\text{ONS}}(K) = \log \Sigma(K) \text{ asympt. simul. cost} \)

\[
\Sigma(K) = \min \text{Tr } T \text{ s.t. } 0 \leq V_x \leq T, \\
\text{Tr } V_x = 1, V_x \leq \Pi_x.
\]
Example: Two-pure-state cq-channel

\[ |\psi_0\rangle = \alpha |0\rangle + \beta |1\rangle \]
\[ |\psi_1\rangle = \alpha |0\rangle - \beta |1\rangle \]

\[ K = \text{span}\{ |\psi_0\rangle, |\psi_1\rangle \} \]

\[ |\psi_0\rangle \langle 0|, \quad |\psi_1\rangle \langle 1| \]

\[ \langle 1|\alpha \rangle > \beta > 0; \quad \alpha^2 + \beta^2 = 1 \]
Example: Two-pure-state cq-channel

\[ 0 \quad \rightarrow \quad |\psi_0\rangle = \alpha |0\rangle + \beta |1\rangle \]

\[ 1 \quad \rightarrow \quad |\psi_1\rangle = \alpha |0\rangle - \beta |1\rangle \]

\[ K = \text{span} \{ |\psi_0\rangle < 01, 1 \psi_0 > < 11^2 \} \]

\[ \langle 1 \rangle \alpha > \beta > 0; \quad \alpha^2 + \beta^2 = 1 \]

\[ \Upsilon(K) = 1, \text{ but } \Upsilon(K \otimes K) \geq \sqrt{\alpha^4 + \beta^4}, \text{ and for } \]

\[ n \text{ large enough, } \Upsilon(K \otimes^n) \geq \sqrt{\alpha^{2n} + \beta^{2n}}. \]
Example: Two-pure-state cq-channel

\[ K = \text{span} \{ |\psi_0 \rangle <0|, \quad 1|\psi_1 \rangle <1| \} \]

\[ |\psi_0 \rangle = \alpha |0\rangle + \beta |1\rangle \]
\[ |\psi_1 \rangle = \alpha |0\rangle - \beta |1\rangle \]

\[ (|1\rangle \alpha > \beta > |0\rangle, \quad \alpha^2 + \beta^2 = 1) \]

\[ \gamma(K) = 1, \text{ but } \gamma(K \otimes K) \geq \sqrt{\alpha^4 + \beta^4}, \text{ and for } n \text{ large enough, } \gamma(K \otimes^n) \geq \sqrt{\alpha^{2n} + \beta^{2n}}. \]

\[ \text{Easy: } A(K) = \sqrt{\alpha^2}, \quad \Sigma(K) = 1 + 2\alpha \beta. \]
5. Lovász number encore

Now the best: Minimize $A(K)$ over all $cg$-channels with the same confusability graph $G (x \sim x' \iff \pi_x \perp \pi_{x'})$. 
Now the best: Minimize $\Lambda(K)$ over all cq-channels with the same confusability graph $G (x \sim x' \iff \Pi_x \neq \Pi_{x'})$.

Thm. $\min \Lambda(K) = \vartheta(G)$; $\min C_{\text{NS}}(K) = \log \vartheta(G)$.

In words: Lovász' number gives the no-signalling assisted capacity of the worst cq-channel with confusability graph $G$.

First capacity interpretation of $\vartheta(G)$ :-)

6. Last words:

- SDP formulas for assisted capacity and simulation cost (one-shot)
- SDP can regularize to a relaxed SDP :-) 
- Capacity interpretation of Lovász number
- Gap between $\mathcal{C}_{\infty}(G)$ and $\log \chi(G)$?
- Regularization necessary? There could be $K$ such that $\chi(K) = \vartheta(G)$ — cf. Ching-Yi Lai’s poster on Monday!
- $\Sigma(G) := \min \{ \Sigma(K) : G \supset K^+K \}$ = ??
- Know only: between $\vartheta(G)$ and $\alpha^*(G)$