No-signalling assisted zero-error communication via quantum channels and the Lovász ϑ number

Andreas Winter (ICREA & UAB Barcelona)
Runyao Duan (UTS Sydney)

arXiv:1409.3426

If you've been partying...

Hungover Summary

1. $C_0(G) \leq \log \vartheta(G)$

2. $C_{0E}(G) \leq \log \vartheta(G)$

3.–5. $C_{0NS}(G) = \log \vartheta(G)$

Hungover Summary

Zero-error capacity of the graph G

Lovász number; it's a semidefinite programme

1. $C_0(G) \leq \log \vartheta(G)$

2. $C_{0E}(G) \leq \log \vartheta(G)$

3.–5. $C_{0NS}(G) = \log \vartheta(G)$

Hungover Summary

1. $C_0(G) \leq \log \vartheta(G)$

2. $C_{0E}(G) \leq \log \vartheta(G)$

3.–5. $C_{0NS}(G) = \log \vartheta(G)$

$C_0(G)$: Zero-error capacity of the graph G

$\vartheta(G)$: Lovász number; it's a semidefinite programme

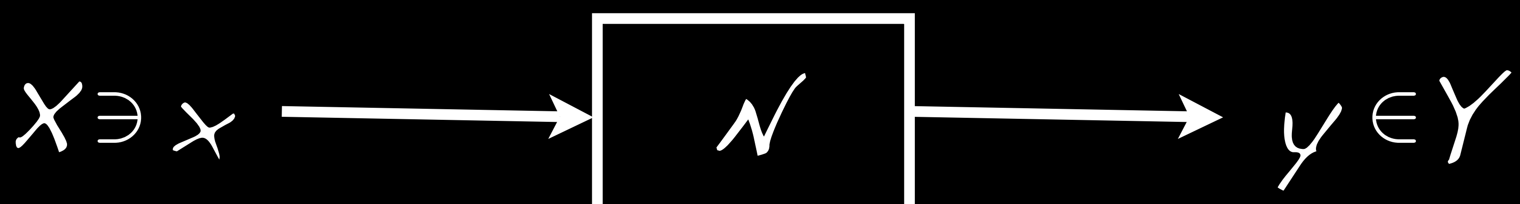
1. ≤ : Can be <

2. ≤ : Might be <

3.–5. = : Yes, it's equality!

1. Channels & graphs

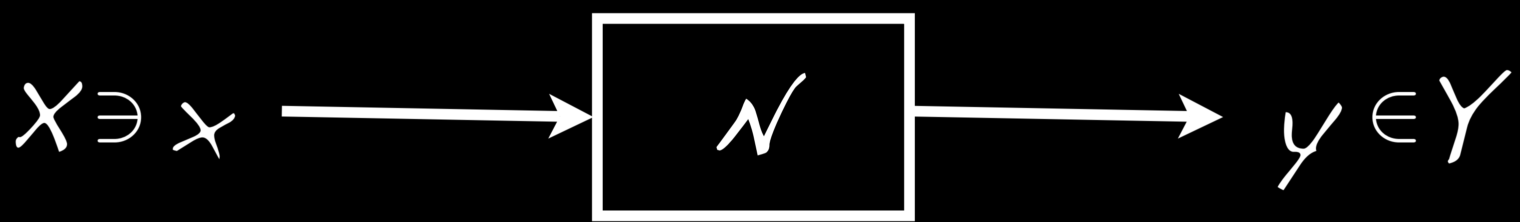
Channel $N : X \longrightarrow Y$, i.e. stochastic map



$N(y|x)$: transition probabilities

1. Channels & graphs

Channel $N : X \longrightarrow Y$, i.e. stochastic map



$N(y|x)$: transition probabilities

Want to send information (in x), such that receiver (seeing y) can be certain about it.

1) Transition graph Γ: bipartite graph on $X \times Y$ with adjacency matrix

$$\Gamma(y|x) = \begin{cases} 1 & \text{if } N(y|x) > 0, \\ 0 & \text{if } N(y|x) = 0. \end{cases}$$

1) Transition graph Γ: bipartite graph on $X \times Y$ with adjacency matrix

$$\Gamma(y|x) = \begin{cases} 1 & \text{if } N(y|x) > 0, \\ 0 & \text{if } N(y|x) = 0. \end{cases}$$

2) Confusability graph G on X: adj. matrix

$$(\mathbb{1}+A)_{xx'} = \begin{cases} 1 & \text{if } N(.|x)^T N(.|x') > 0, \\ 0 & \text{if } N(.|x)^T N(.|x') = 0. \end{cases}$$

1) Transition graph Γ: bipartite graph on $X \times Y$ with adjacency matrix

$$\Gamma(y|x) = \begin{cases} 1 & \text{if } N(y|x) > 0, \\ 0 & \text{if } N(y|x) = 0. \end{cases}$$

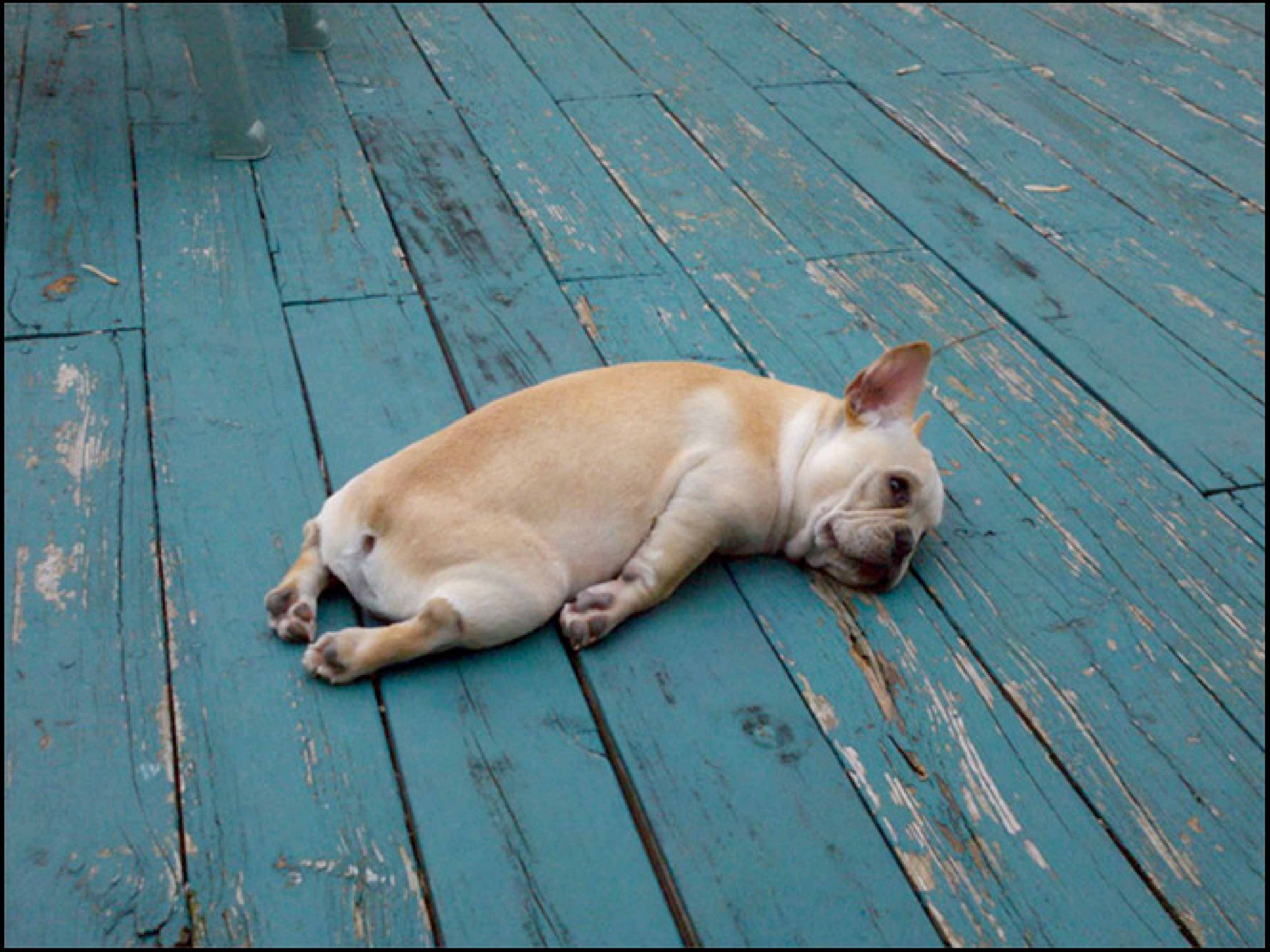
2) Confusability graph G on X: adj. matrix

$$(\mathbb{1}+A)_{xx'} = \begin{cases} 1 & \text{if } N(\cdot|x)^T N(\cdot|x') > 0, \\ 0 & \text{if } N(\cdot|x)^T N(\cdot|x') = 0. \end{cases}$$

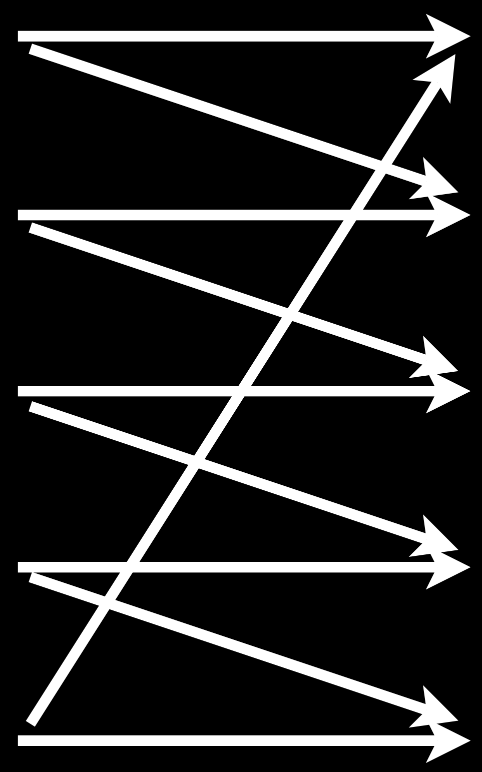
Lovász convention:

$x \sim x'$ iff $x = x'$ or xx' edge

Example?

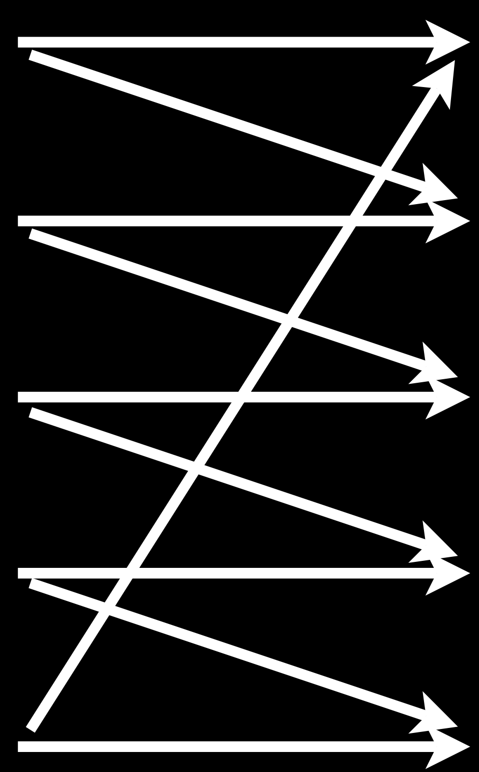


$\Gamma = T_5$



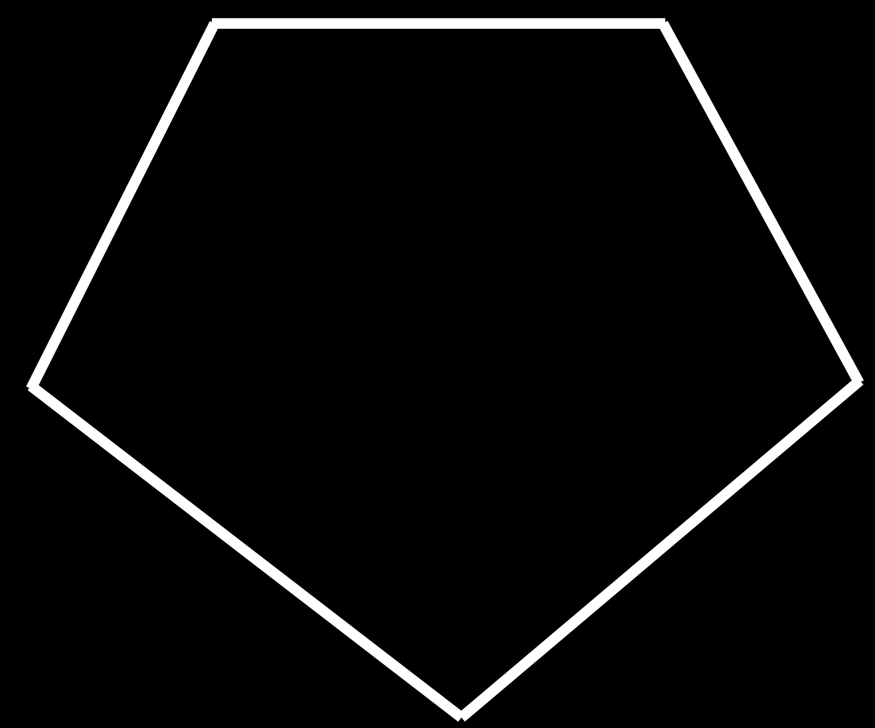
typewriter channel

$\Gamma = T_5$



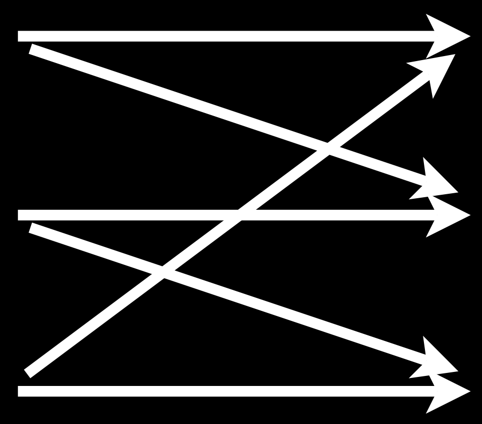
typewriter channel

$G = C_5$

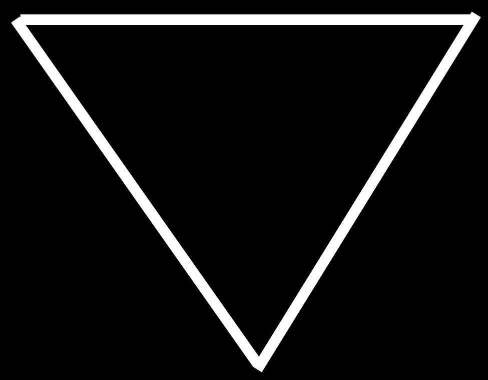


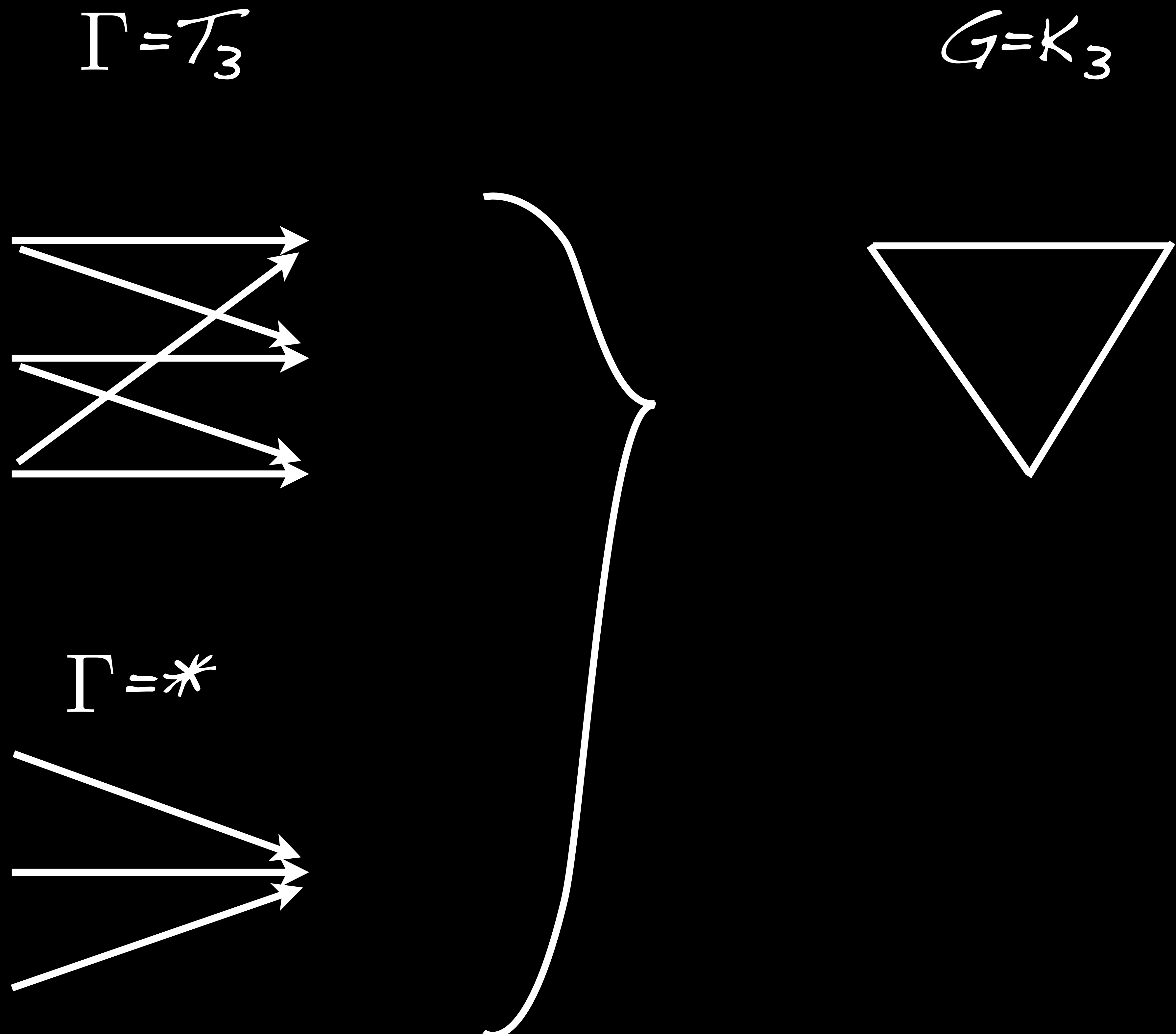
pentagon

$\Gamma = T_3$



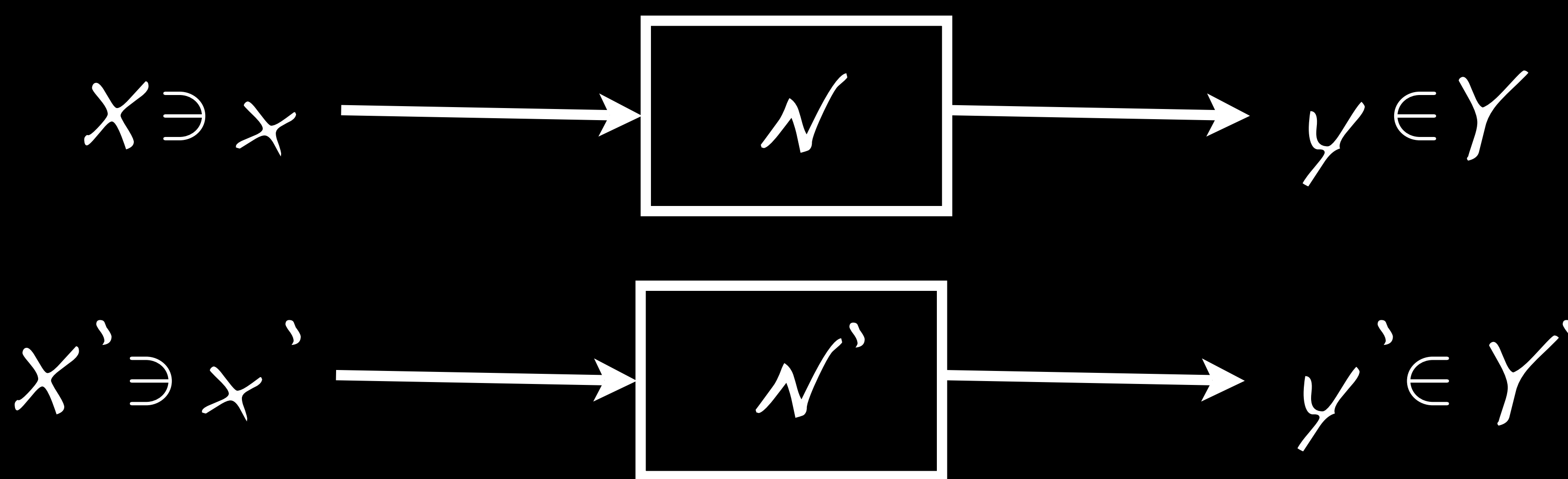
$G = K_3$




Γ=T3
Γ=*
G=K3

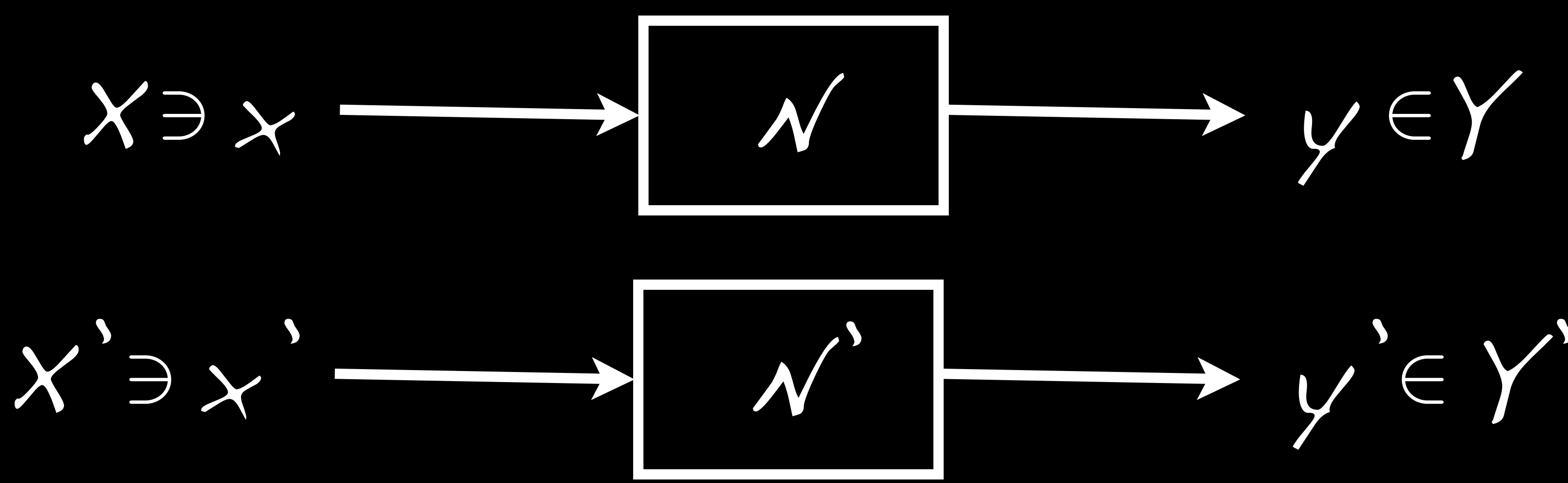
Product channels:

$$N \times N'(yy'|xx') = N(y|x)N'(y'|x')$$



Product channels:

$$\mathcal{N}\times\mathcal{N}'(yy'|xx') = \mathcal{N}(y|x)\mathcal{N}'(y'|x')$$



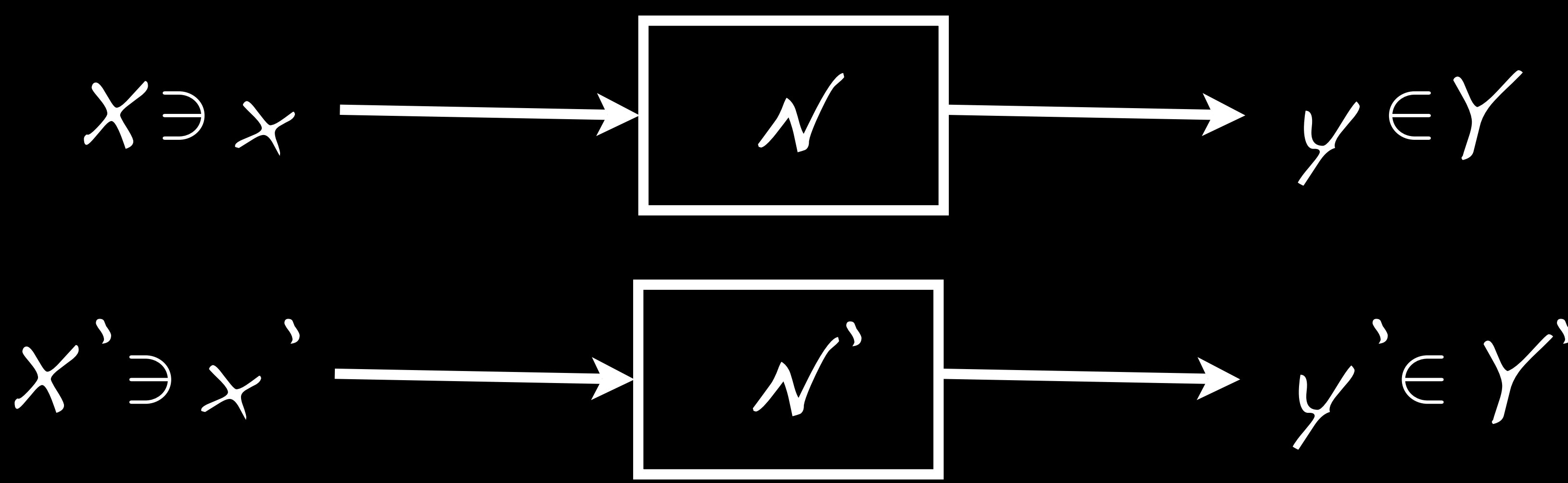
Graphs via Kronecker/tensor product:

$$\Gamma(\mathcal{N}\times\mathcal{N}') = \Gamma\otimes\Gamma'$$

$$\mathbb{1}+A(\mathcal{N}\times\mathcal{N}') = (\mathbb{1}+A)\otimes(\mathbb{1}+A')$$

Product channels:

$$N \times N'(yy'|xx') = N(y|x)N'(y'|x')$$



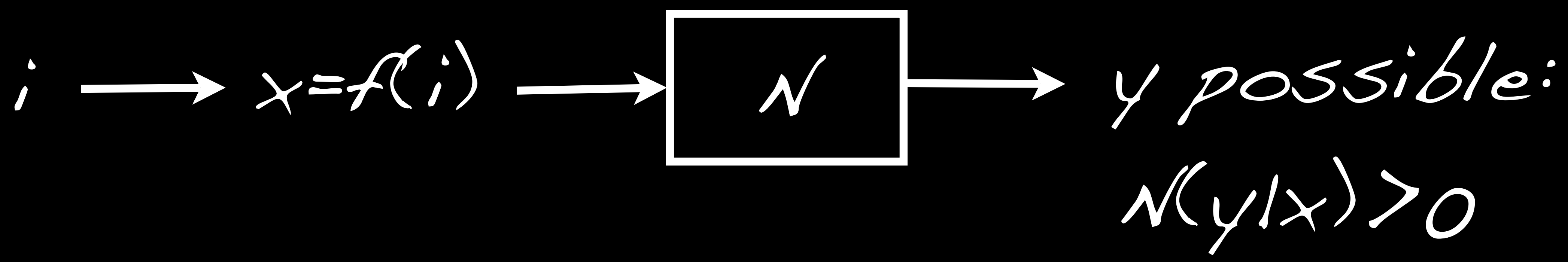
Graphs via Kronecker/tensor product:

$$\Gamma(N \times N') = \Gamma \otimes \Gamma'$$

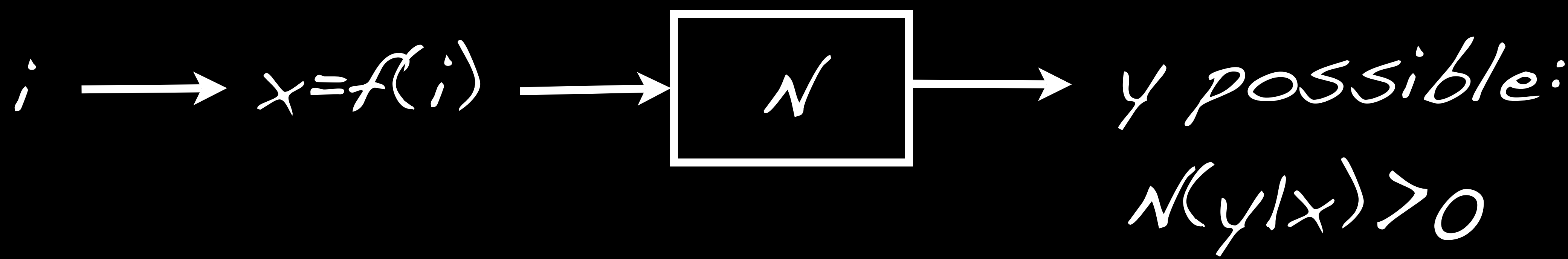
$$\mathbb{1} + A(N \times N') = (\mathbb{1} + A) \otimes (\mathbb{1} + A')$$

Strong graph product $G \times G'$

$1\frac{1}{2}$. Zero-error transmission



$1\frac{1}{2}$. Zero-error transmission

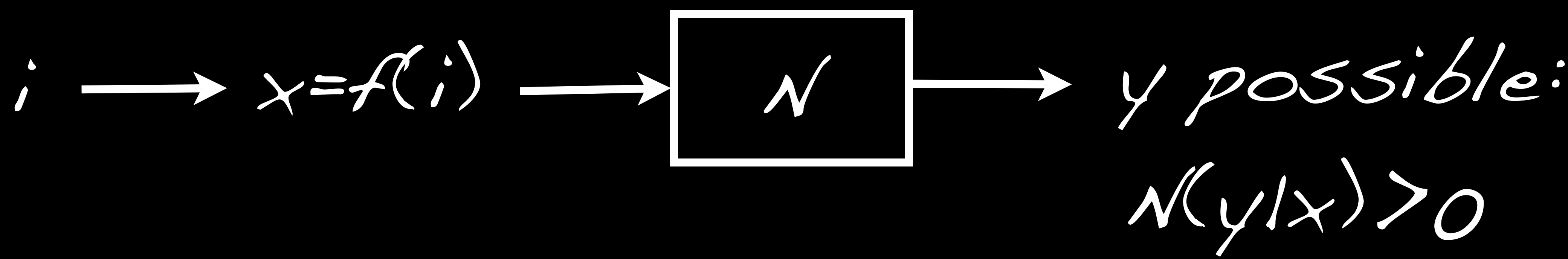


Hence: codebook $\{f(i)\} \subset X$ must be an independent set in G.

Maximum size:

$\alpha(G) :=$ independence number of G.

$1\frac{1}{2}$. Zero-error transmission



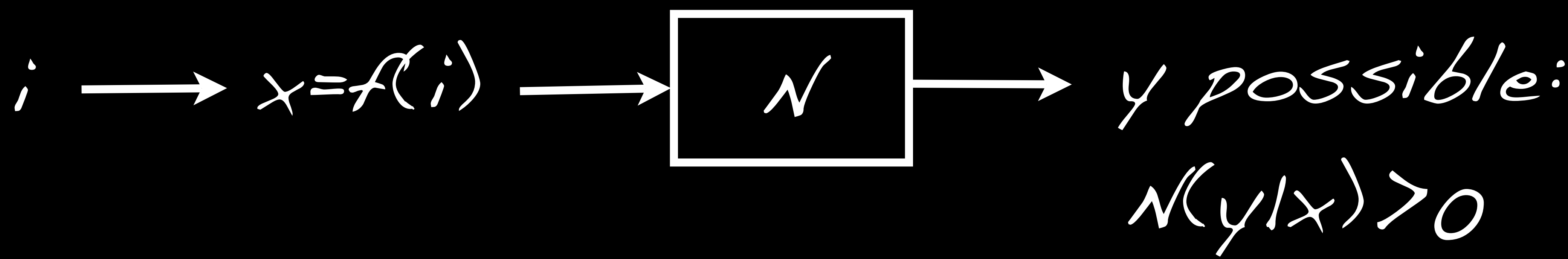
Hence: codebook $\{f(i)\} \subset X$ must be an independent set in G.

Maximum size:

$\alpha(G)$:= independence number of G.

Well-known to be NP-complete!

$1\frac{1}{2}$. Zero-error transmission



Hence: codebook $\{f(i)\} \subset X$ must be an independent set in G.

Maximum size:

$\alpha(G)$:= independence number of G.

Well-known to be NP-complete!

Upper bounds!?

$$\alpha(G) \leq \vartheta(G) = \max \operatorname{Tr} BJ \text{ s.t. } B \geq 0,\ \operatorname{Tr} B = 1,$$

$$B_{xy} = 0 \ \forall xy \in G.$$

[L. Lovász, IEEE-IT 25(1):1-7, 1979]

$$\alpha(G) \leq \vartheta(G) = \max \operatorname{Tr} BJ \text{ s.t. } B \geq 0,\ \operatorname{Tr} B = 1,$$
$$B_{xy} = 0\ \forall xy \in G.$$

[L. Lovász, IEEE-IT 25(1):1-7, 1979]

$$\leq \alpha^*(\Gamma) = \max \sum_x \omega_x \text{ s.t. } \omega_x \geq 0 \ \&$$
$$\forall y \sum_x \Gamma(y|x)\omega_x \leq 1.$$

[C.E. Shannon, IRE-IT 2(3):8-19, 1956]

$$\alpha(G) \le \vartheta(G) = \max \operatorname{Tr} BJ \text{ s.t. } B \ge 0,\ \operatorname{Tr} B = 1,$$
$$B_{xy} = 0 \ \forall xy \in G.$$

[L. Lovász, IEEE-IT 25(1):1-7, 1979]

$$\le \alpha^*(\Gamma) = \max \sum_x \omega_x \text{ s.t. } \omega_x \ge 0 \ \&$$
$$\forall y \sum_x \Gamma(y|x)\omega_x \le 1.$$

[C.E. Shannon, IRE-IT 2(3):8-19, 1956]

Best: $\alpha^*(G) = \min \alpha^*(\Gamma)$ s.t. $G \supset$ graph of Γ

$\alpha(G) \leq \vartheta(G) = \max \mathrm{Tr}\, BJ$ s.t. $B \geq 0$, $\mathrm{Tr}\, B = 1$,

$B_{xy} = 0\ \forall xy \in G$.

[L. Lovász, IEEE-IT 25(1):1-7, 1979]

$\leq \alpha^*(\Gamma) = \max \sum_x \omega_x$ s.t. $\omega_x \geq 0$ &

$\forall y \sum_x \Gamma(y|x)\omega_x \leq 1$.

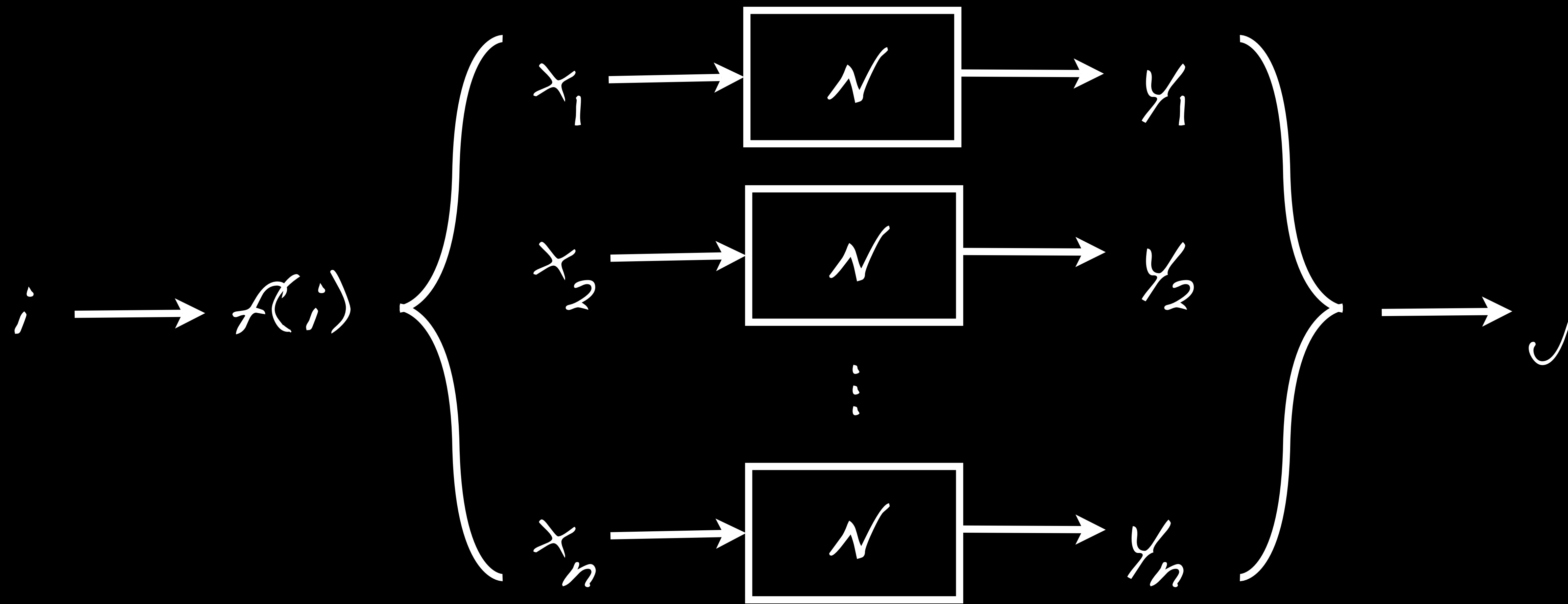
[C.E. Shannon, IRE-IT 2(3):8-19, 1956]

Best: $\alpha^*(G) = \min \alpha^*(\Gamma)$ s.t. $G \supset$ graph of Γ

(Attained at Γ that has an output for every maximal clique of G: $\Gamma(C|x)=1$ iff $x \in C$.)

[C.E. Shannon, IRE-IT 2(3):8-19, 1956]

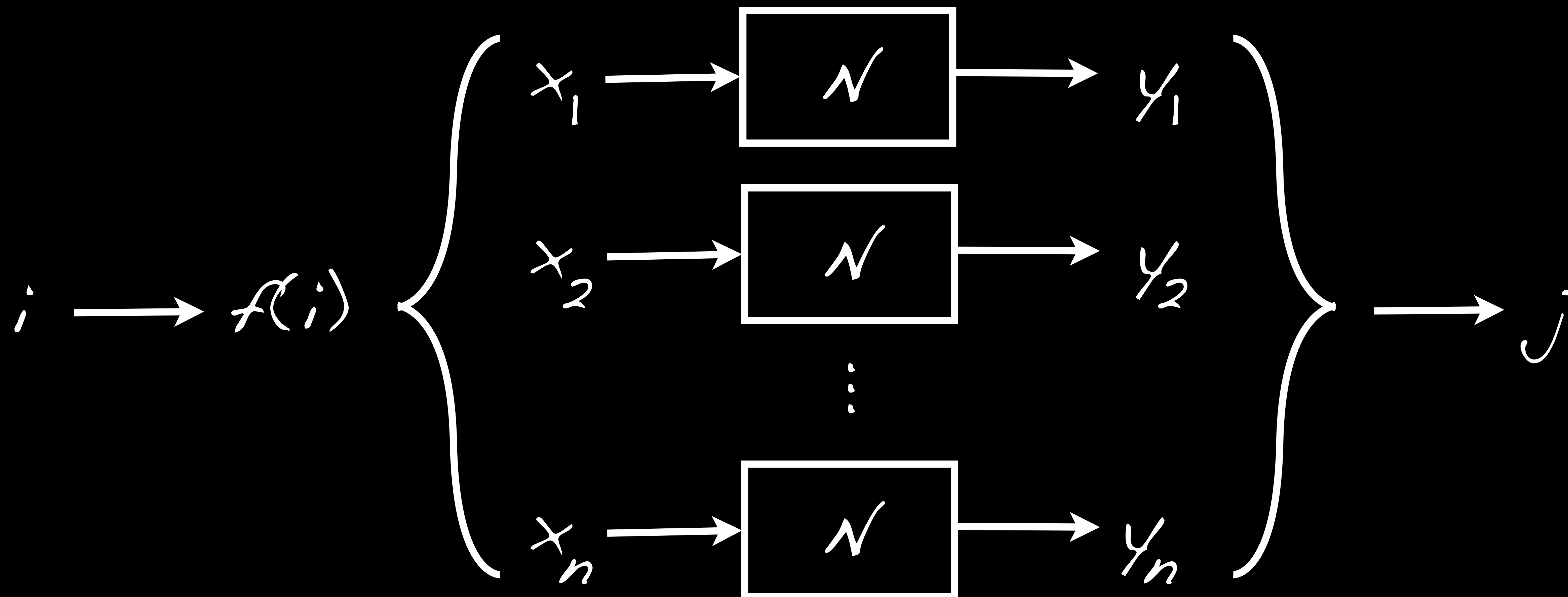
Asymptotically many channel uses - capacity:



$$C_0(G) = \lim \frac{1}{n} \log \alpha(G^{\times n})$$

[C.E. Shannon, IRE-IT 2(3):8-19, 1956]

Asymptotically many channel uses - capacity:



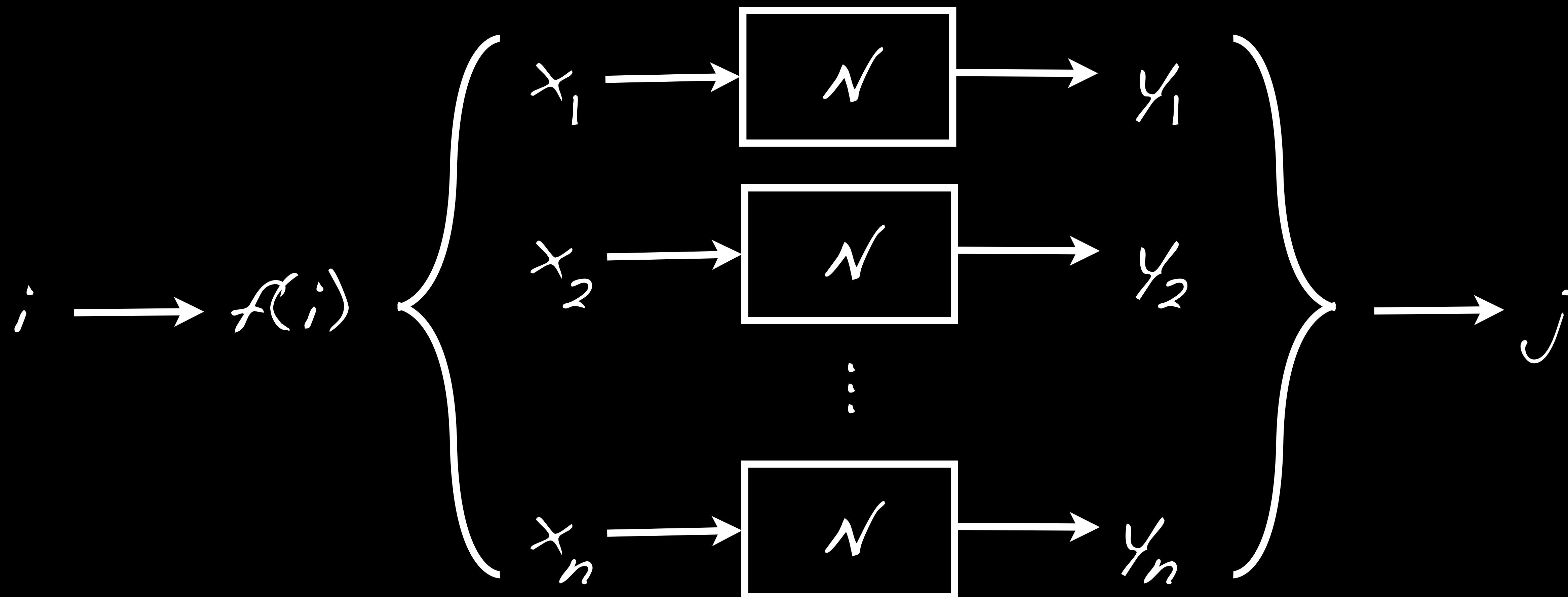
$$C_0(G) = \lim \frac{1}{n} \log \alpha(G^{\times n})$$

=sup because

$$\alpha(G \times H) \geq \alpha(G)\alpha(H)$$

[C.E. Shannon, IRE-IT 2(3):8-19, 1956]

Asymptotically many channel uses – capacity:



$$C_0(G) = \lim \frac{1}{n} \log \alpha(G^{\times n}) \leq \log \vartheta(G)$$

=sup because

$\alpha(G \times H) \geq \alpha(G)\,\alpha(H)$

$\vartheta(G \times H) = \vartheta(G)\,\vartheta(H)$!

[L. Lovász, IEEE-IT 25(1):1-7, 1979]

$$\log \alpha(G) \leq C_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma)$$

Also fractional packing number multiplicative:

$$\alpha^*(\Gamma \otimes \Gamma') = \alpha^*(\Gamma)\alpha^*(\Gamma'),$$

$$\alpha^*(G \times H) = \alpha^*(G)\alpha^*(H) \;!$$

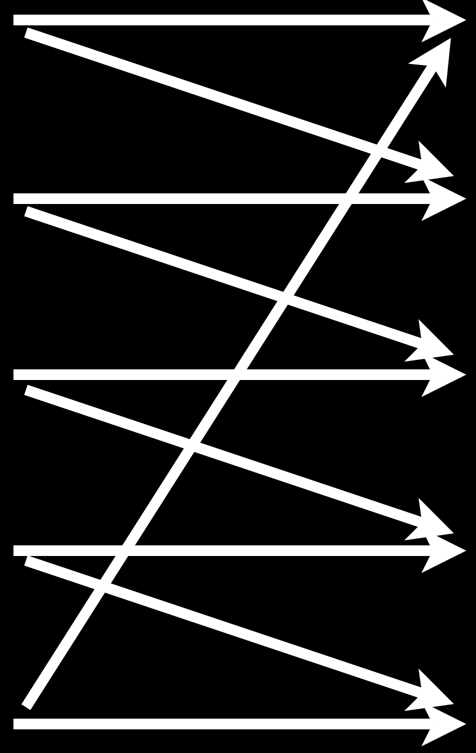
$$\log \alpha(G) \leq C_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma)$$

All inequalities can be strict; first and last:

Ex. Typewriter channel/pentagon

$\alpha(C_5)=2$, $\alpha(C_5 \times C_5)=5>4$,

but $\vartheta(C_5)=\sqrt{5}$, and $\alpha^*(T_5) = 5/2$.



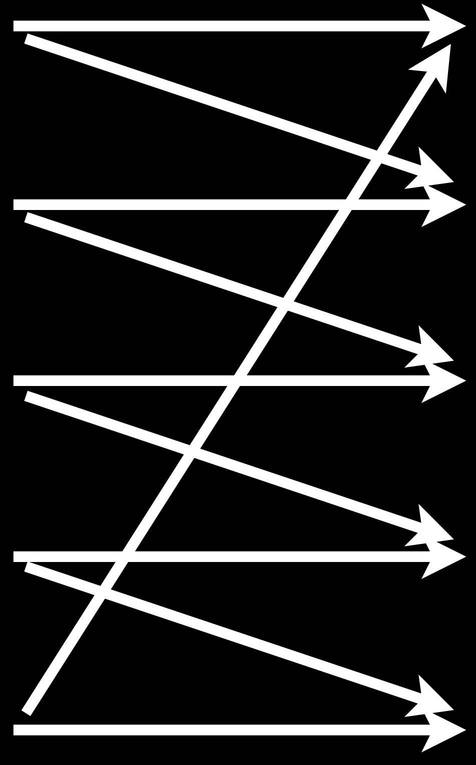
$\log \alpha(G) \leq C_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma)$

All inequalities can be strict; first and last:

Ex. Typewriter channel/pentagon

$\alpha(C_5)=2$, $\alpha(C_5 \times C_5)=5>4$,

but $\vartheta(C_5)=\sqrt{5}$, and $\alpha^*(T_5) = 5/2$.



Note: $\alpha^*(T_3) = 3/2$, but $\alpha^*(*) = 1!$

$\log \alpha(G) \leq C_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma)$

All inequalities can be strict; first and last:

Random graphs $G \sim G(n, 1/2)$ have, whp,

$\alpha(G) \approx \log n$, $\vartheta(G) \approx \sqrt{n}$, $\alpha^*(G) \approx n/(\log n)$

$$\log \alpha(G) \le C_0(G) \le \log \vartheta(G) \le \log \alpha^*(\Gamma)$$

All inequalities can be strict; middle due to W. Haemers [IEEE-IT 25(2):231-232, 1979], via a different algebraic and multiplicative bound on α which sometimes(!) is better than ϑ.

$$\log \alpha(G) \leq C_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma)$$

All inequalities can be strict; middle due to W. Haemers [IEEE-IT 25(2):231-232, 1979], via a different algebraic and multiplicative bound on α which sometimes(!) is better than ϑ.

However: w/o sacrificing multiplicativity, ϑ cannot be improved [Acín/Duan/Sainz/AW, 2014].

$$\log \alpha(G) \leq C_0(G) \leq \log \vartheta(G) \leq \log \alpha^*(\Gamma)$$

All inequalities can be strict; middle due to W. Haemers [IEEE-IT 25(2):231-232, 1979], via a different algebraic and multiplicative bound on α which sometimes(!) is better than ϑ.

Determination of $C_0(G)$ open, not even known to be computable...

[N. Alon/E. Lubetzky, IEEE-IT 52(5):2172-2176, 2006]

$$C_0(G) \leq \log \vartheta(G)$$

Idea: Perhaps we can close the gap by allowing additional resources in the en-/decoding?

$$C_0(G) \le \log \vartheta(G)$$

Idea: Perhaps we can close the gap by allowing additional resources in the en-/decoding?

+ feedback [C.E. Shannon, IRE-IT 2(3):8-19, 1956]

$C_{0F}(\Gamma) = \log \alpha^*(\Gamma)$, with constant activating noiseless bits.

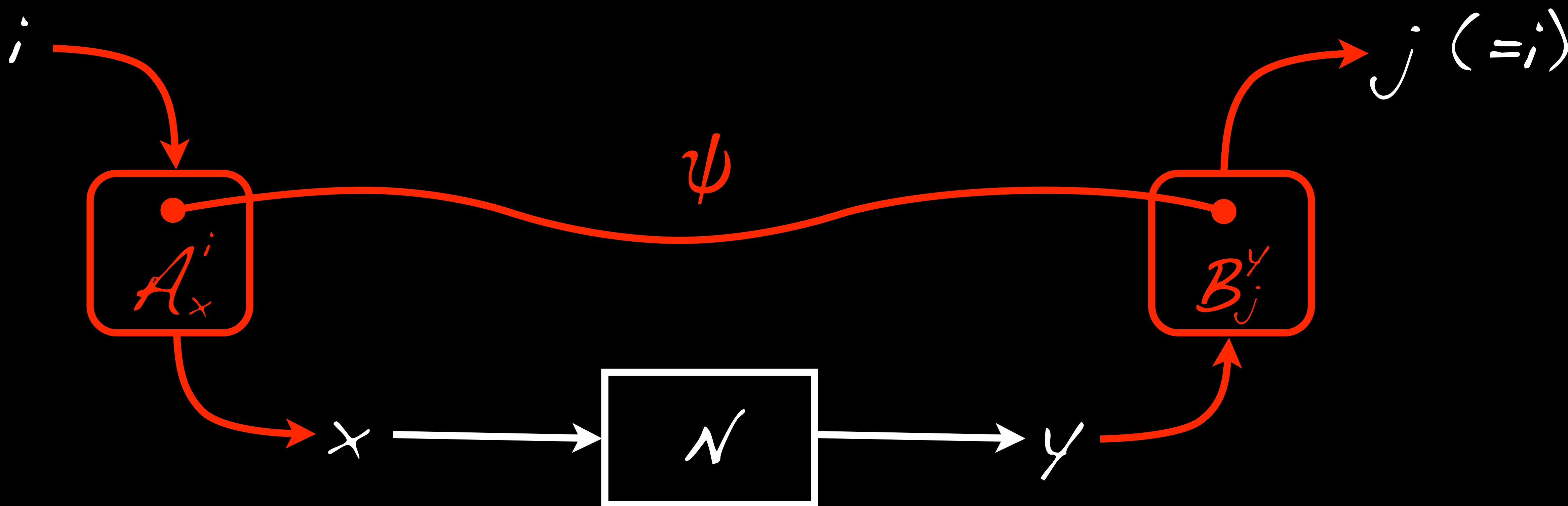
$$C_0(G) \leq \log \vartheta(G)$$

Idea: Perhaps we can close the gap by allowing additional resources in the en-/decoding?

+ feedback [C.E. Shannon, IRE-IT 2(3):8-19, 1956]
+ entanglement (quantum correlations)
+ no-signalling correlations

2. Free non-local resources

For instance, with free entanglement:



Maximum #messages =: $\tilde{\alpha}(G)$

Can show that this depends only on G;

furthermore can be > $\alpha(G)$...

[T.S. Cubitt et al., PRL 104:230503, 2010]

Known: $\alpha(G) \leq \tilde{\alpha}(G) \leq \vartheta(G)$

[S. Beigi, PRA 82:010303, 2010;
R. Duan/S. Severini/AW,
IEEE-IT 59(2):1164-1174, 2013.]

Known: $\alpha(G) \le \tilde{\alpha}(G) \le \vartheta(G)$

Since ϑ is multiplicative under strong graph product, $\vartheta(G \times H) = \vartheta(G)\vartheta(H)$, get:

$$C_0(G) \le C_{0E}(G) = \lim \frac{1}{n} \log \tilde{\alpha}(G^{\times n}) \le \log \vartheta(G)$$

Known: $\alpha(G) \leq \tilde{\alpha}(G) \leq \vartheta(G)$

Since ϑ is multiplicative under strong graph product, $\vartheta(G \times H) = \vartheta(G)\vartheta(H)$, get:

$$C_0(G) \leq C_{0E}(G) = \lim \frac{1}{n} \log \tilde{\alpha}(G^{\times n}) \leq \log \vartheta(G)$$

Known examples of separation

[D. Leung/L. Mancinska/W. Matthews/+2, CMP 311:97-111, 2012; J. Briët/H. Buhrman/D. Gijswijt, PNAS 110:19227, 2012]

Known: $\alpha(G) \le \tilde{\alpha}(G) \le \vartheta(G)$

Since ϑ is multiplicative under strong graph product, $\vartheta(G \times H) = \vartheta(G)\vartheta(H)$, get:

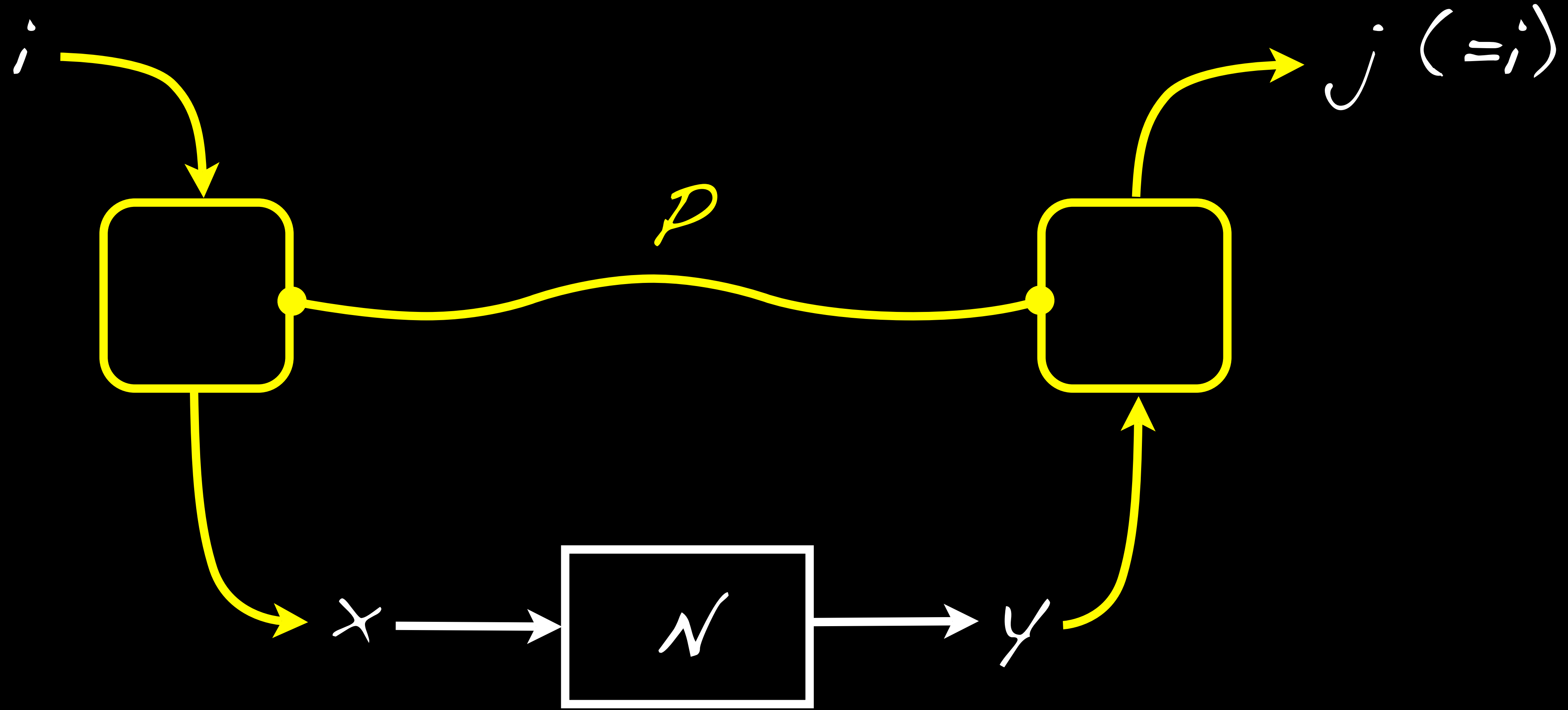
$$C_0(G) \le C_{0E}(G) = \lim \frac{1}{n} \log \tilde{\alpha}(G^{\times n}) \le \log \vartheta(G)$$

Unknown whether = or < !

Known examples of separation

[D. Leung/L. Mancinska/W. Matthews/+2, CMP 311:97–111, 2012; J. Briët/H. Buhrman/D. Gijswijt, PNAS 110:19227, 2012]

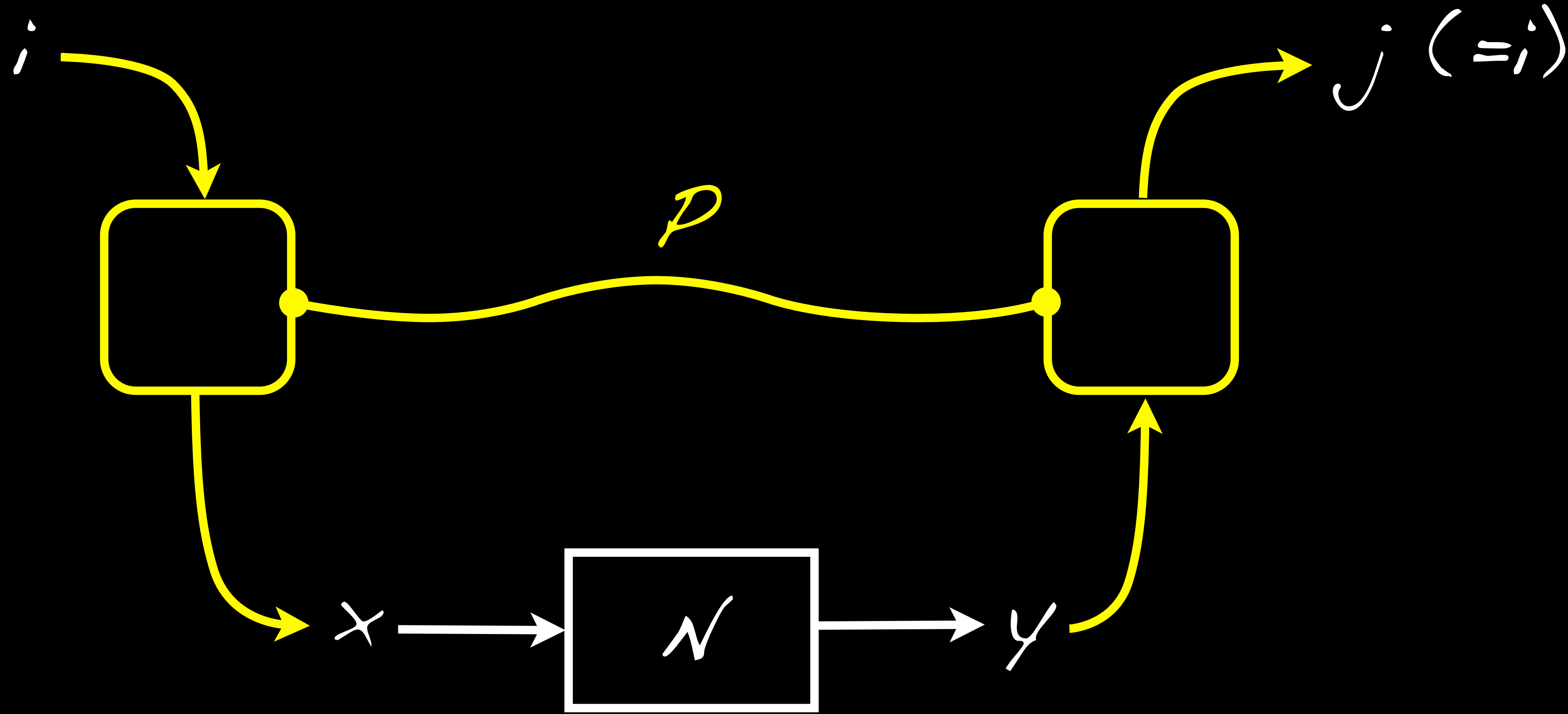
Allowing general no-signalling correlation:



This is no-signalling assisted zero-error code if $j=i$ with probability 1.

I.e., for all $j \neq i$ & edges xy in Γ, $P(xj|iy)=0$

Allowing general no-signalling correlation:



This is no-signalling assisted zero-error code if $j=i$ with probability 1.

I.e., for all $j \neq i$ & edges xy in Γ, $P(xj|iy)=0$

Maximum #msg. with $P \in NS =: \overline{\alpha}(\Gamma)$

$$\overline{\alpha}(\Gamma) = \max m \text{ s.t. } P(xj|iy) \in NS,\ i,j=1\ldots m,$$
$$\forall i\neq j\ \forall xy\in\Gamma\quad P(xj|iy)=0.$$

$$\overline{\alpha}(\Gamma) = \max m \text{ s.t. } P(xj|iy) \in NS,\ i,j=1\ldots m,$$
$$\forall i\neq j\ \forall xy \in \Gamma\ \ P(xj|iy)=0.$$

Clear: Can test given m efficiently by linear programming. Less obvious:

$$\overline{\alpha}(\Gamma) = \max m \text{ s.t. } P(xj|iy) \in NS,\ i,j=1\ldots m,$$
$$\forall i\neq j\ \forall xy \in \Gamma \quad P(xj|iy)=0.$$

Clear: Can test given m efficiently by linear programming. Less obvious:

Thm. $\overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$, with α^* the fractional packing number of Γ:

$$\alpha^*(\Gamma) = \max \sum_x \omega_x \text{ s.t. } \omega_x \geq 0 \text{ \& for all } y,$$
$$\sum_x \Gamma(y|x)\omega_x \leq 1.$$

[T.S. Cubitt et al., IEEE-IT 57(8):5509-5523, 2011]

Thm. $\overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$, with α^* the fractional packing number of Γ:

$$\alpha^*(\Gamma) = \max \sum_x w_x \text{ s.t. } w_x \geq 0 \text{ \& for all } y,$$
$$\sum_x \Gamma(y|x) w_x \leq 1.$$

[T.S. Cubitt et al., IEEE-IT 57(8):5509-5523, 2011]

Cor. Due to multiplicativity of α^*,

$$C_{ONS}(\Gamma) = \lim \frac{1}{n} \log \overline{\alpha}(\Gamma^{\otimes n}) = \log \alpha^*(\Gamma).$$

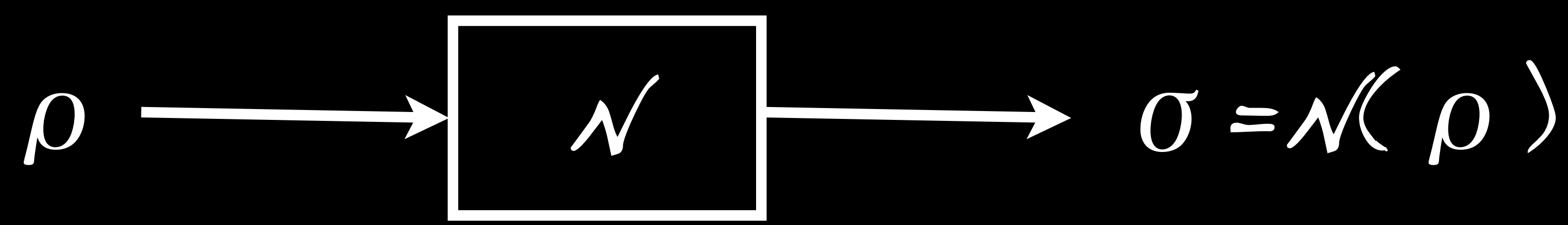
[C.E. Shannon, 1956: Same answer for feedback-assisted capacity!]

...so this is too big - what now?!

3. Quantum version...

3. Quantum version...

Should really consider quantum channels $\mathcal{N}: B(A) \longrightarrow B(B)$, cptp map on states:



Kraus form: $\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger, \; \sum_i E_i^\dagger E_i = \mathbb{1}$

For quantum channel (cptp map) $N: B(A) \longrightarrow B(B)$, with Kraus op's E_i:

Define $K = \text{span}\{E_i\} \subset B(A \to B)$ and $S = K^\dagger K = \text{span}\{E_i^\dagger E_j\} \subset B(A)$ as natural analogues of the transition and confusability graphs.

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013;
R. Duan/AW, arXiv:1409.3426]

For quantum channel (cptp map) $N: B(A) \longrightarrow B(B)$, with Kraus op's E_i:

Define $K = \text{span}\{E_i\} \subset B(A \to B)$ and $S = K^\dagger K = \text{span}\{E_i^\dagger E_j\} \subset B(A)$ as natural analogues of the transition and confusability graphs.

($S = S^\dagger \ni \mathbb{1}$, so S is an operator system)

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013;
R. Duan/AW, arXiv:1409.3426]

Define $K = \mathrm{span}\{E_i\} \subset B(A \to B)$ and

$S = K^\dagger K = \mathrm{span}\{E_i^\dagger E_j\} \subset B(A)$.

For classical channel, Kraus operators

are $\propto \Gamma(y|x)\, |y\rangle\langle x|$, so:

$K = \mathrm{span}\{\Gamma(y|x)\, |y\rangle\langle x|\} \quad \leftrightarrow \quad \Gamma,$

$S = \mathrm{span}\{|x'\rangle\langle x| \text{ s.t. } x \sim x'\} \quad \leftrightarrow \quad G.$

...hence S, K extend G, Γ to quantum...

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013;
R. Duan/AW, arXiv:1409.3426]

Define $K = \mathrm{span}\{E_i\} \subset B(A \to B)$ and

$S = K^\dagger K = \mathrm{span}\{E_i^\dagger E_j\} \subset B(A)$.

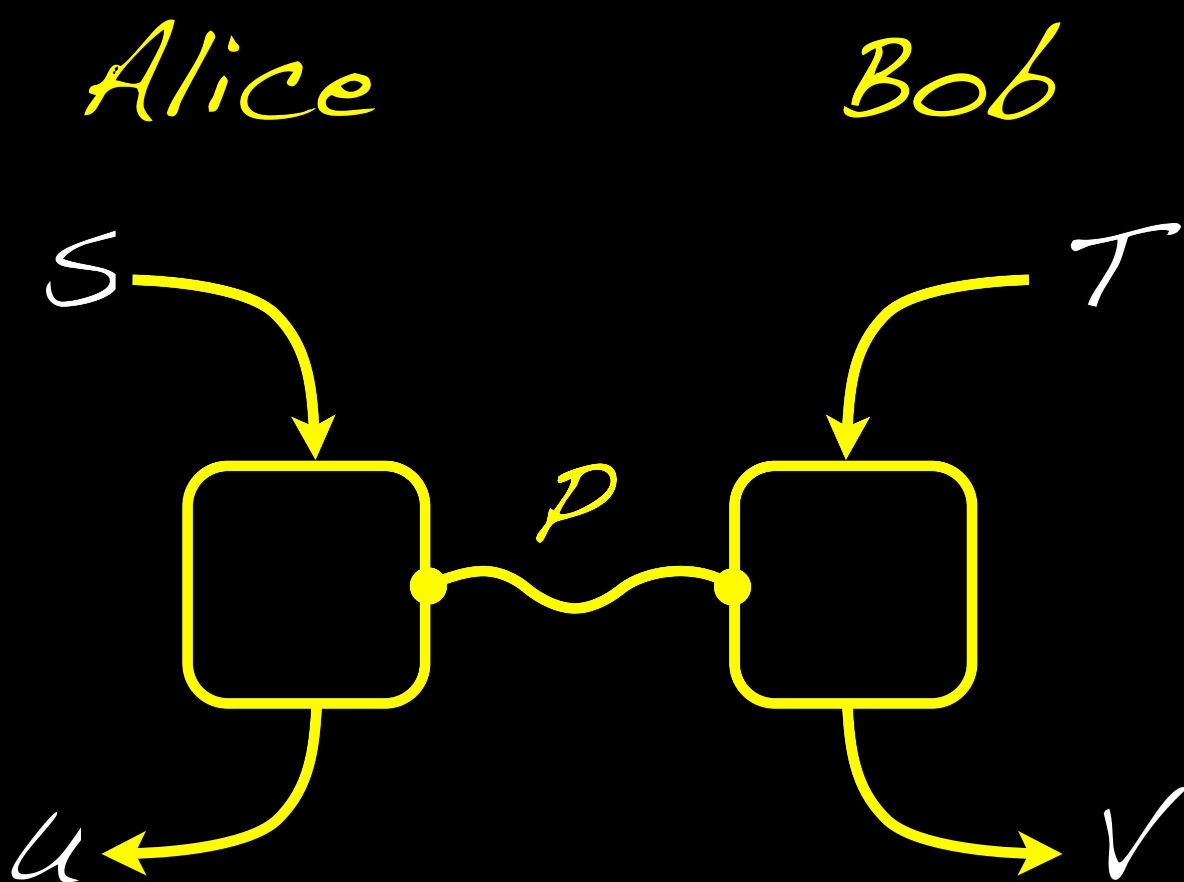
Can show: Zero-error transmission assisted by entanglement (or without) depends only on S.

Below treat assistance by quantum no-signalling correlations, which will turn out to depend only on K.

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013; R. Duan/AW, arXiv:1409.3426]

Quantum no-signalling:

Cf. W. Matthews' talk on Wed!



$P: S \otimes T \to U \otimes V$ cptp

No-signalling means:

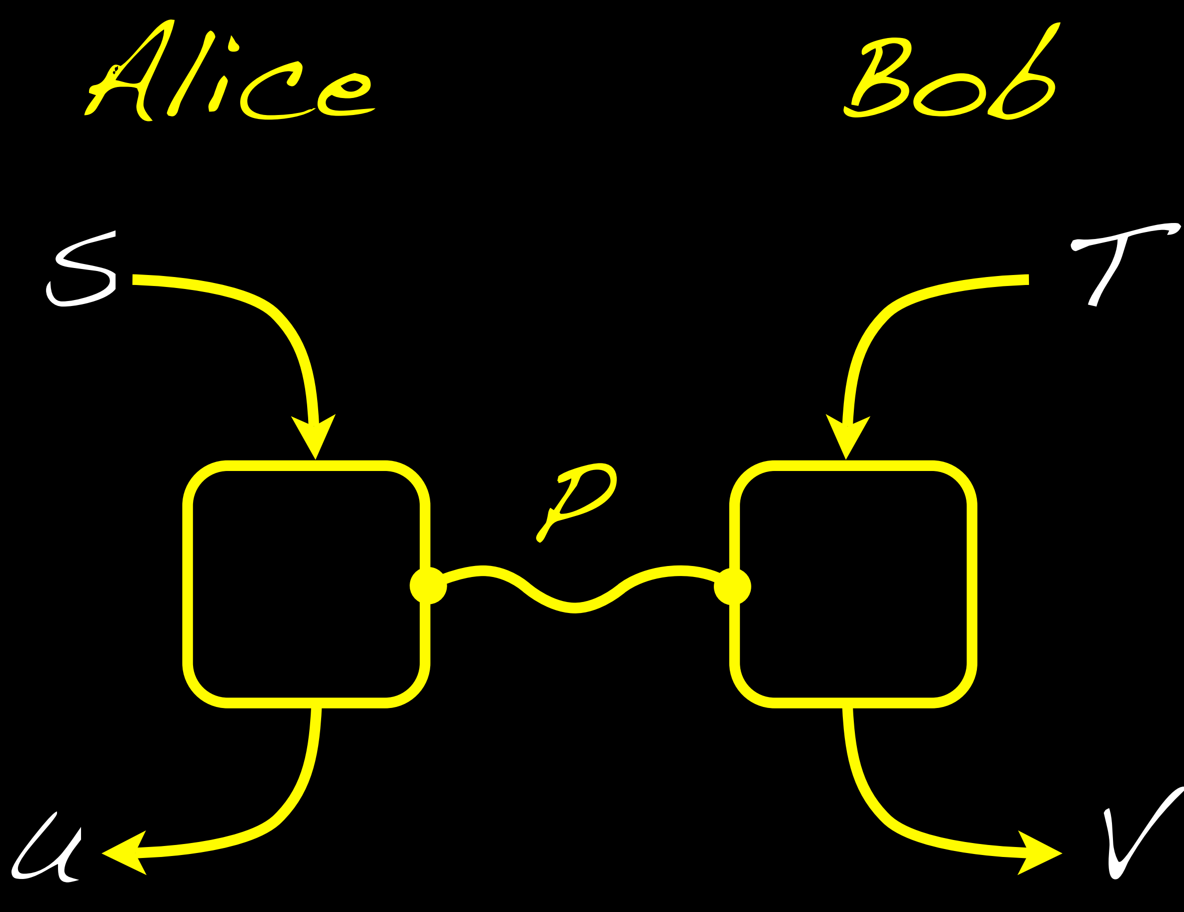
$$\mathrm{Tr}_U\, P(\sigma \otimes \tau) = B(\tau),$$

$$\mathrm{Tr}_V\, P(\sigma \otimes \tau) = A(\sigma).$$

[D. Beckman et al., PRA 64:052309, 2001; T. Eggeling et al., Europhy. Lett. 57(6):782-788, 2002; M. Piani et al., PRA 74:012305, 2006]

Quantum no-signalling:

Cf. W. Matthews' talk on Wed!



$P: S \otimes T \to U \otimes V$ cptp

No-signalling means:

$$\mathrm{Tr}_U\, P(\sigma \otimes \tau) = B(\tau),$$

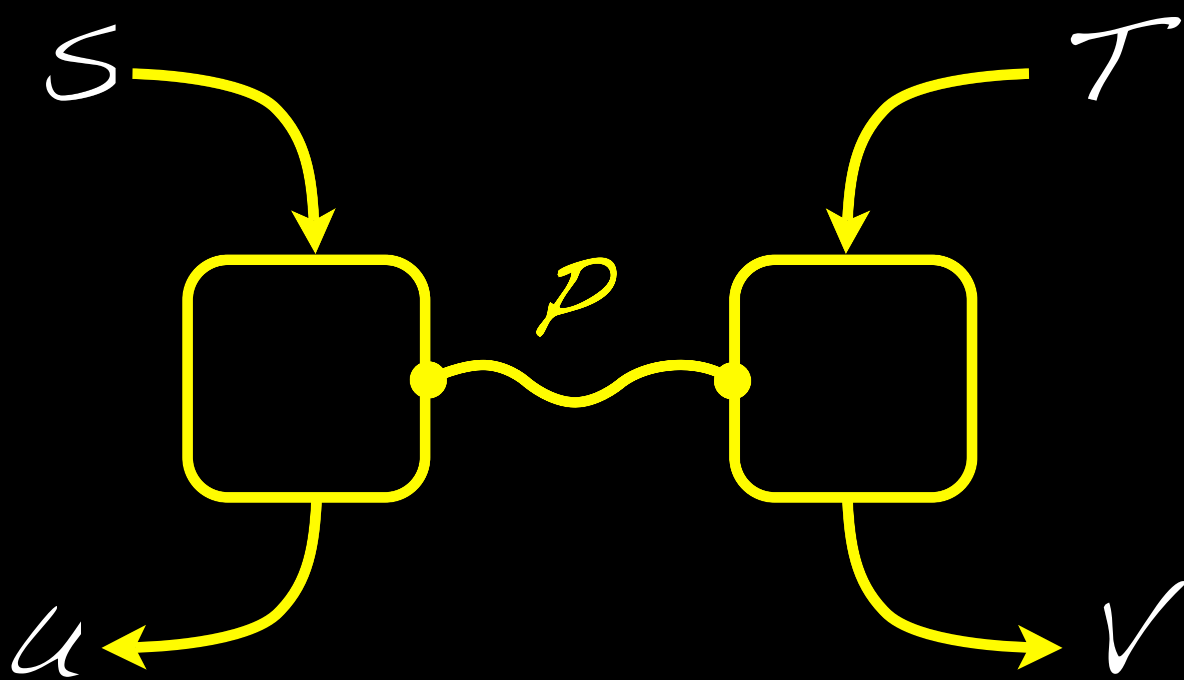
$$\mathrm{Tr}_V\, P(\sigma \otimes \tau) = A(\sigma).$$

Equiv.: P linear combination of $A_i \otimes B_i$ plus semidef. constraint for "cptp"

[D. Beckman et al., PRA 64:052309, 2001; T. Eggeling et al., Europhy. Lett. 57(6):782-788, 2002; M. Piani et al., PRA 74:012305, 2006]

Quantum no-signalling:

Cf. W. Matthews' talk on Wed!



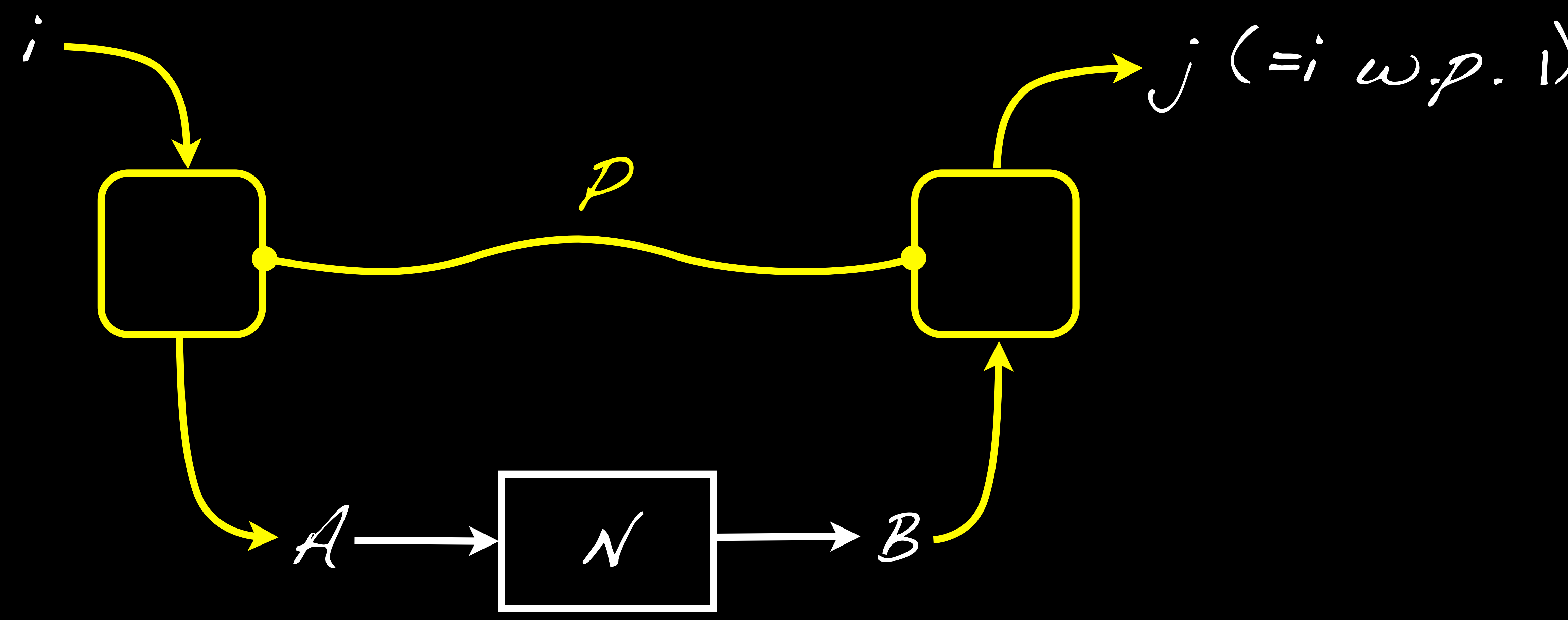
$P: S \otimes T \to U \otimes V$ cptp

$$Tr_U \, P(\sigma \otimes \tau) = B(\tau),$$

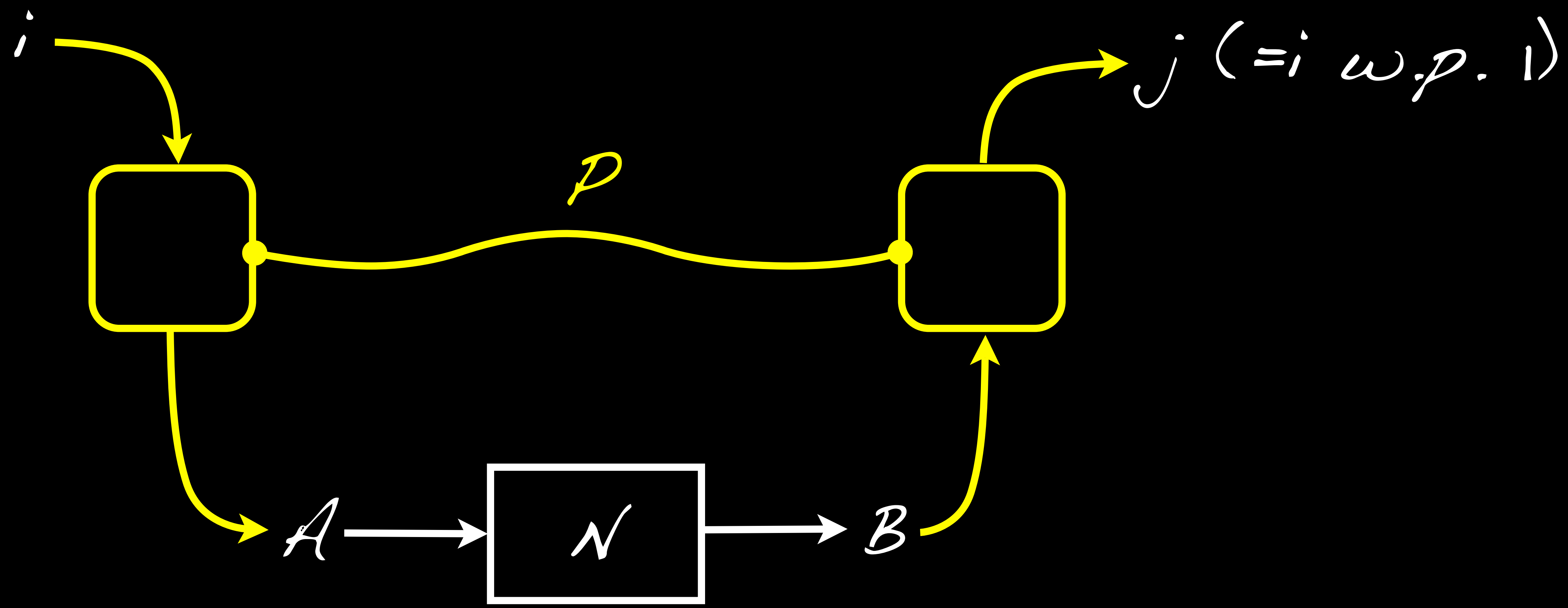
$$Tr_V \, P(\sigma \otimes \tau) = A(\sigma).$$

Although formally a channel with two simultaneous inputs, the no-signalling condition ensures that Alice can use her "box" without waiting. Bob is left with a conditional channel...

No-signalling assisted communication:

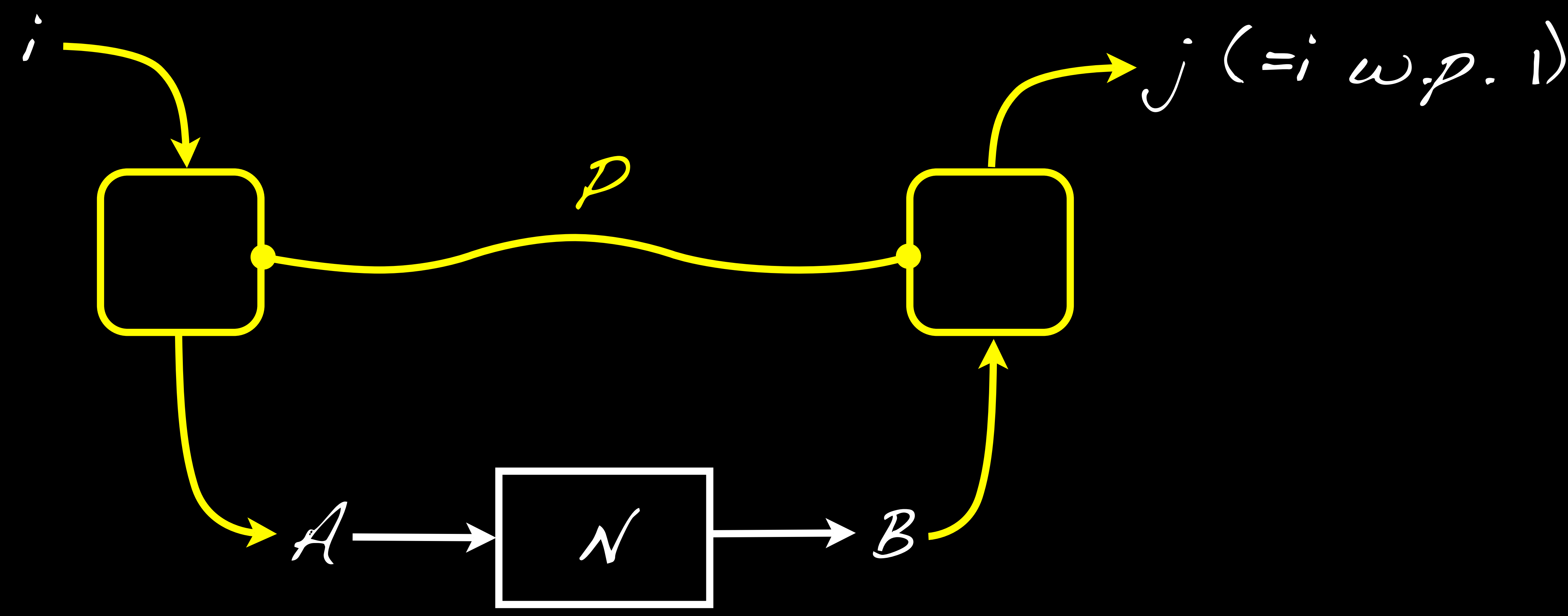


No-signalling assisted communication:



Maximum #msg. with $P \in NS =: \overline{\alpha}(K)$

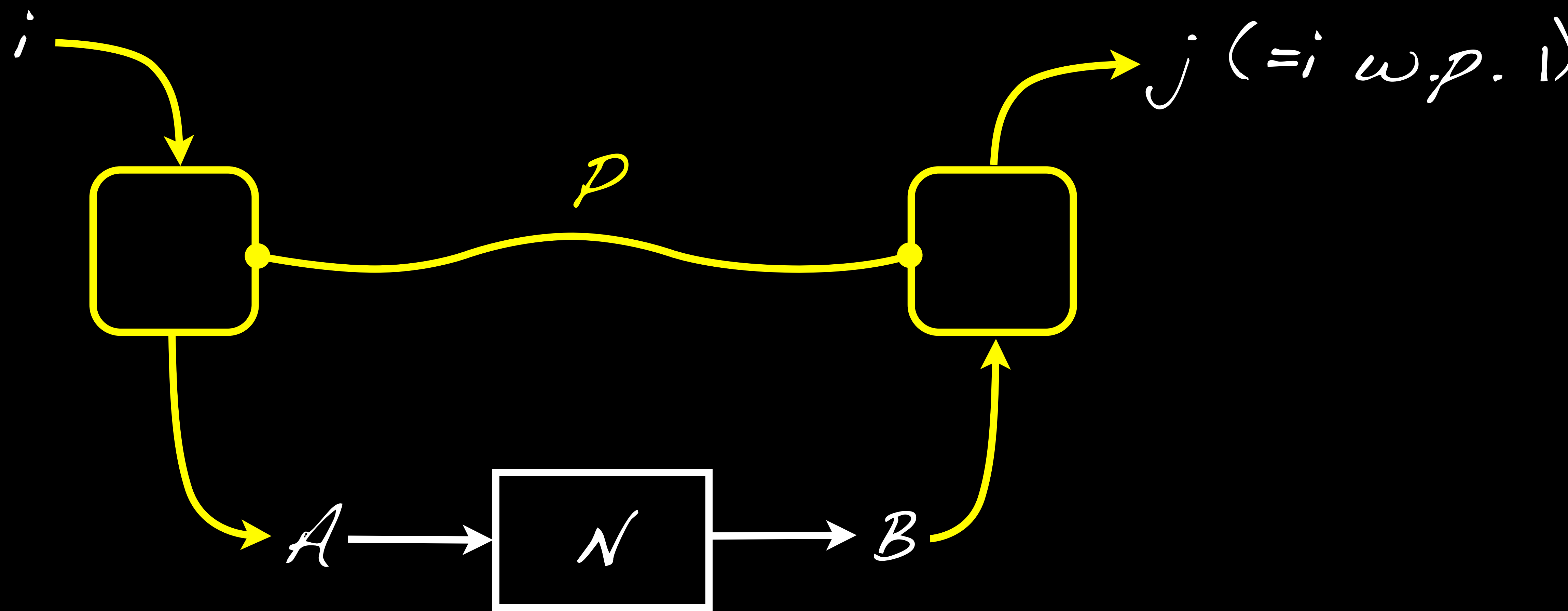
No-signalling assisted communication:



Maximum #msg. with $P \in NS =: \overline{\alpha}(K)$

Similar definitions of $\alpha(S)$ and $\tilde{\alpha}(S)$ via max. # of messages; all reducing to previous notions for classical channels.

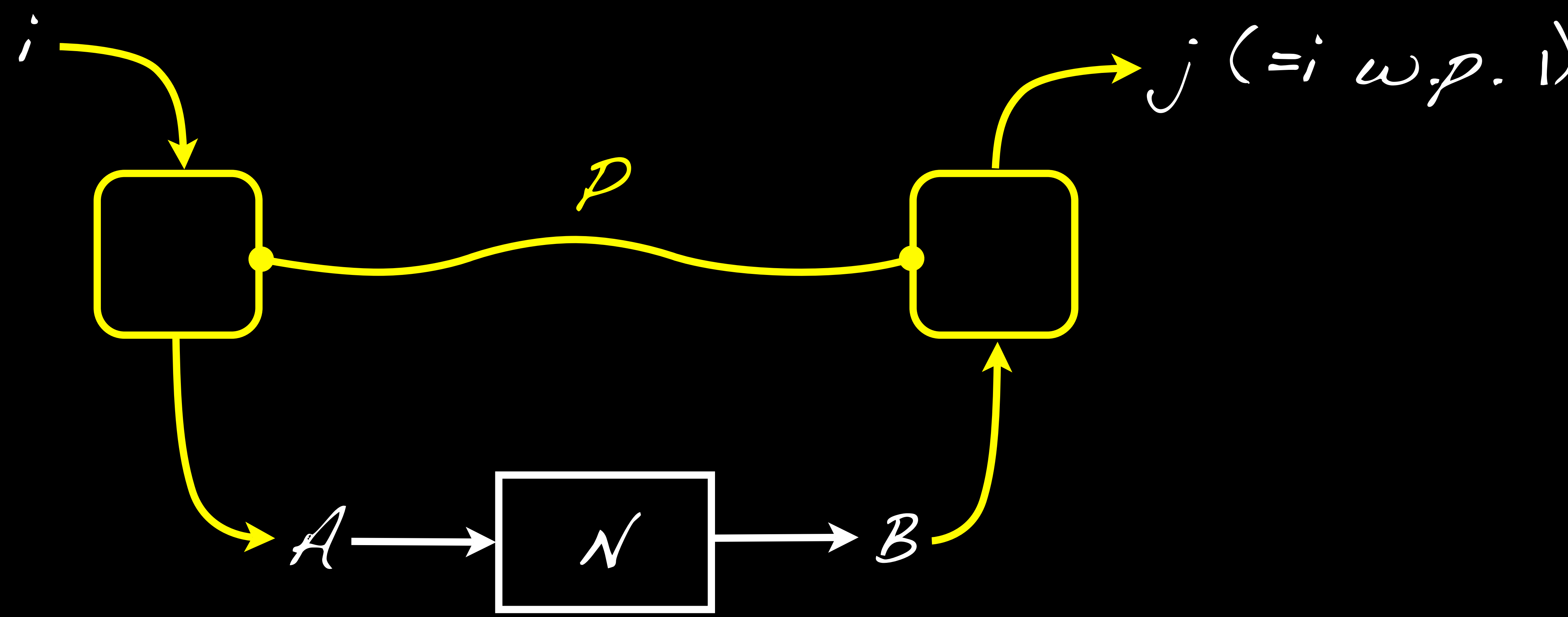
No-signalling assisted communication:



Thm. [RD/AW] $\overline{\alpha}(\mathcal{K}) = \lfloor \Upsilon(\mathcal{K}) \rfloor$, where

$$\Upsilon(\mathcal{K}) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes \mathbb{1},$$
$$\operatorname{Tr}_A U = \mathbb{1}^B,$$
$$\Pi(S \otimes \mathbb{1} - U) = 0.$$

No-signalling assisted communication:



Thm. [RD/AW] $\overline{\alpha}(\mathcal{K}) = \lfloor \Upsilon(\mathcal{K}) \rfloor$, where

$$\Upsilon(\mathcal{K}) = \max \operatorname{Tr} S \text{ s.t. } 0 \le U^{AB} \le S \otimes \mathbb{1},$$

$$\operatorname{Tr}_A U = \mathbb{1}^B,$$

$$\Pi(S \otimes \mathbb{1} - U) = 0.$$

Π: support projection of Choi matrix of $\mathcal{N}$

$$\Upsilon(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes \mathbb{1},$$
$$\operatorname{Tr}_A U = \mathbb{1}^B,$$
$$\Pi(S \otimes \mathbb{1} - U) = 0.$$

...reduces to classical fractional packing number for classical channel.

[T.S. Cubitt et al., IEEE-IT 57(8):5509-5523, 2011]

$$\Upsilon(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes \mathbb{1},$$
$$\operatorname{Tr}_A U = \mathbb{1}^B,$$
$$\Pi(S \otimes \mathbb{1} - U) = 0.$$

...reduces to classical fractional packing number for classical channel.

However, in general much more complex; for instance not multiplicative, i.e. $\Upsilon(K \otimes K') \geq \Upsilon(K)\Upsilon(K')$, sometimes strict.

$$\Upsilon(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes \mathbb{1},$$
$$\operatorname{Tr}_A U = \mathbb{1}^B,$$
$$\Pi(S \otimes \mathbb{1} - U) = 0.$$

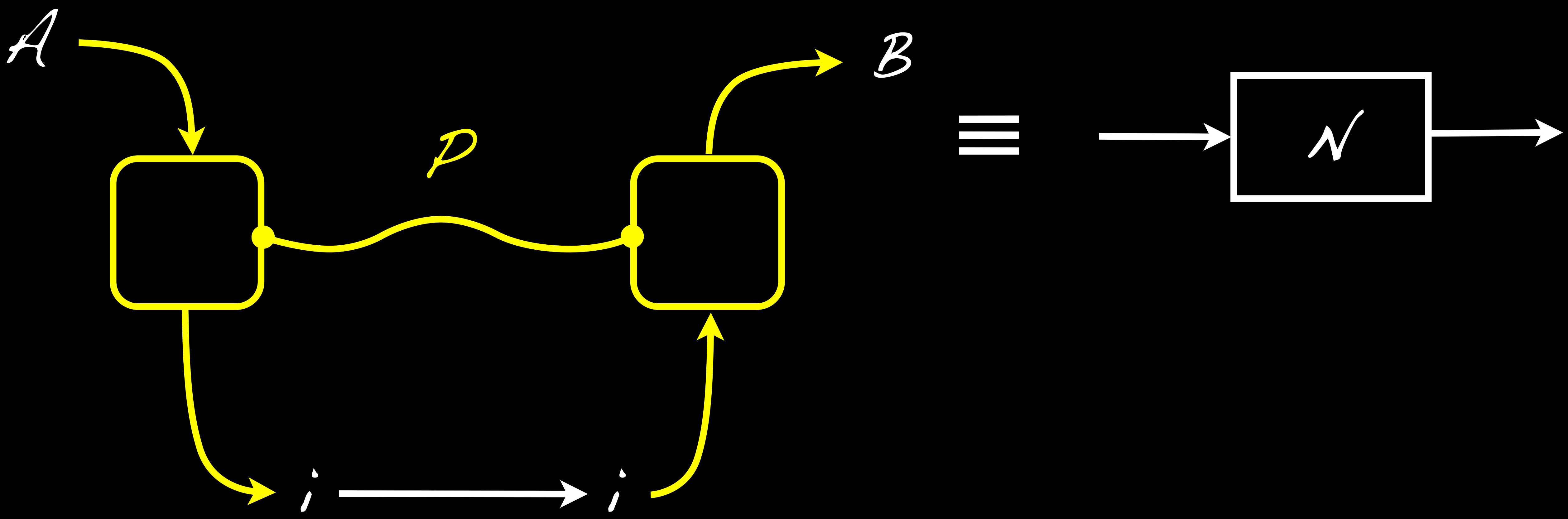
...reduces to classical fractional packing number for classical channel.

However, in general much more complex; for instance not multiplicative, i.e.

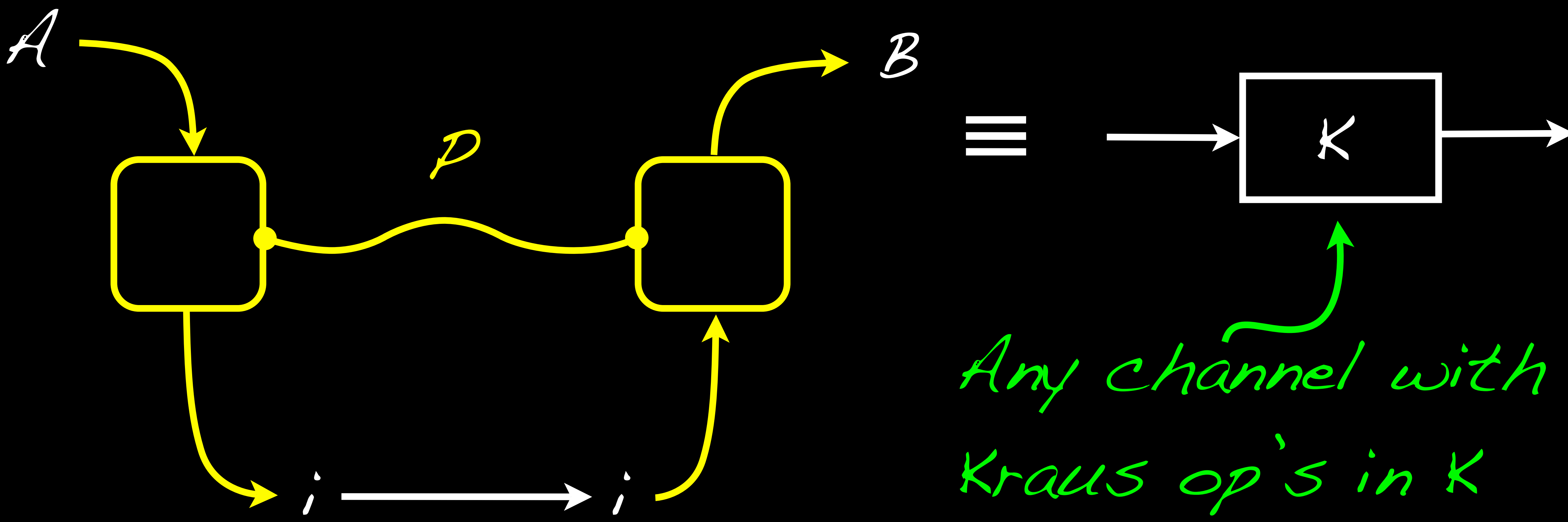
$\Upsilon(K \otimes K') \geq \Upsilon(K)\Upsilon(K')$, sometimes strict.

What is $C_{ONS}(K) = \lim \frac{1}{n} \log \Upsilon(K^{\otimes n})$?

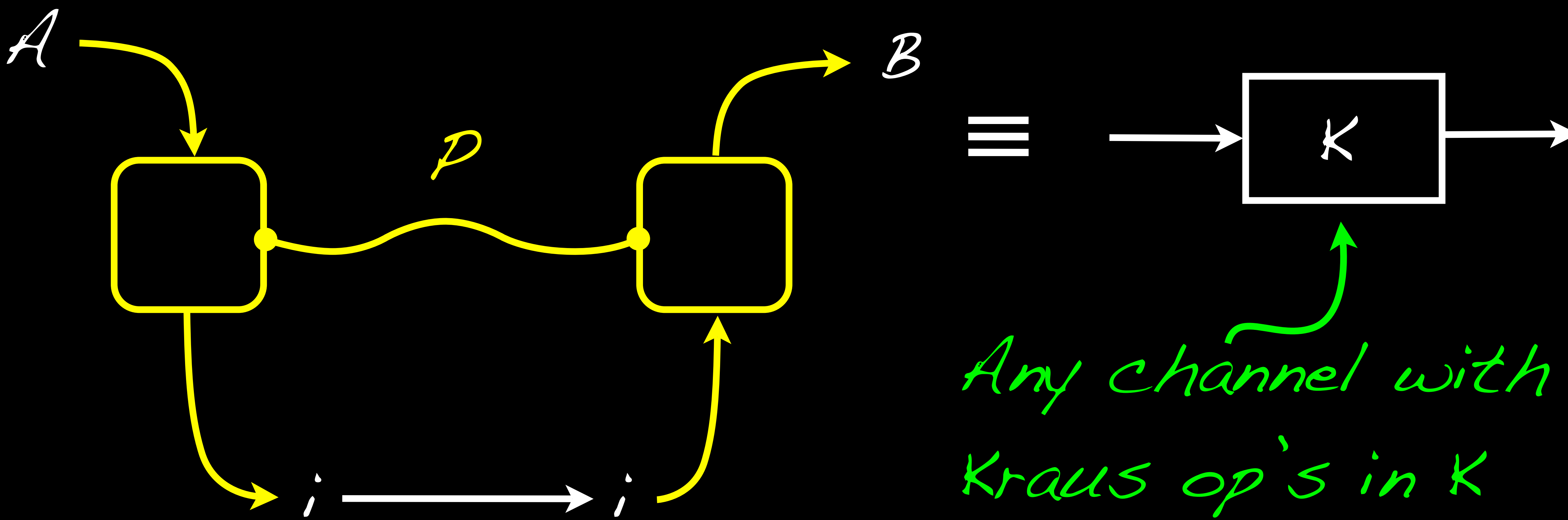
No-signalling assisted channel simulation:



No-signalling assisted channel simulation:



No-signalling assisted channel simulation:

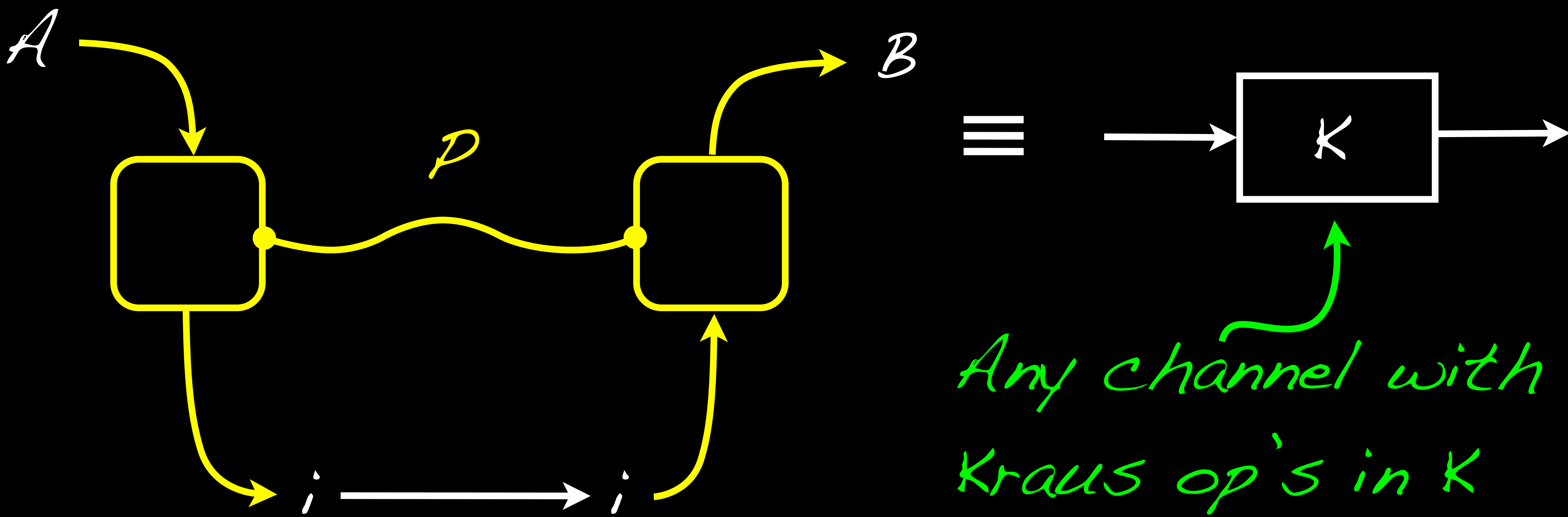


Thm. [RD/AW] Min #msg $= \lceil \Sigma(K) \rceil$, w/

$$\Sigma(K) = \min \operatorname{Tr} T \text{ s.t. } 0 \leq V^{AB} \leq \mathbb{1} \otimes T,$$

$$\operatorname{Tr}_B V = \mathbb{1}^A, \ \Pi^{\perp} V = 0.$$

No-signalling assisted channel simulation:



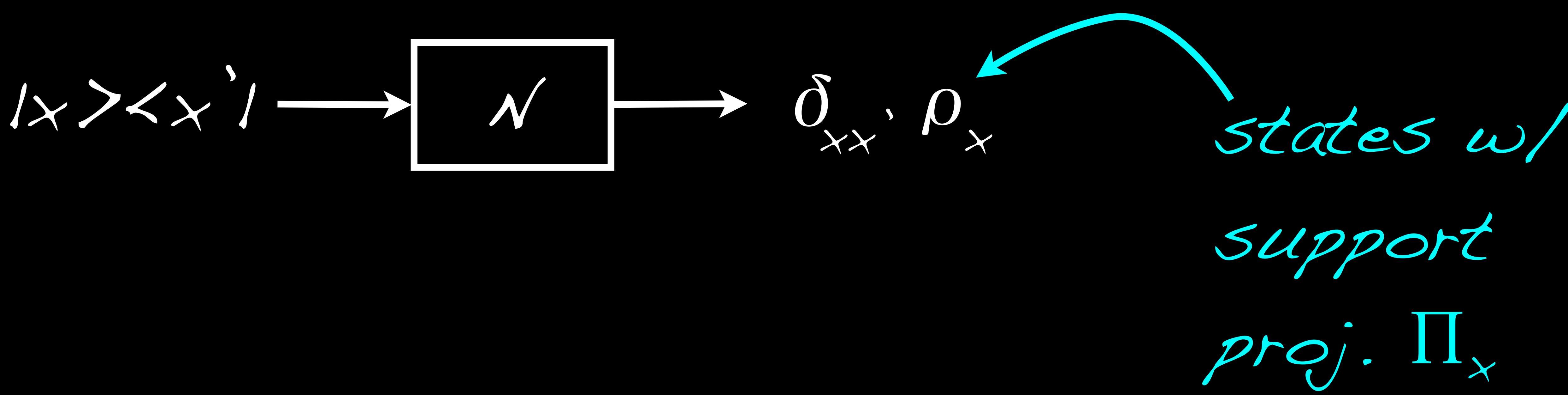
Thm. [RD/AW] Min #msg $= \lceil \Sigma(K) \rceil$, w/

$$\Sigma(K) = \min \operatorname{Tr} T \text{ s.t. } 0 \leq V^{AB} \leq \mathbb{1} \otimes T,$$

$$\operatorname{Tr}_B V = \mathbb{1}^A, \ \Pi^\perp V = 0.$$

...reduces to $\alpha^*(\Gamma)$ for classical channels.

4. Cq-channels



These are still "very classical", e.g. have confusability graph G, $x \sim x'$ iff $\Pi_x \Pi_{x'} \neq 0$.

4. Cq-channels

states w/ support proj. Π_x

These are still "very classical", e.g. have confusability graph G, $x \sim x'$ iff $\Pi_x \Pi_{x'} \neq 0$.

$\Pi = \sum_x |x\rangle\langle x| \otimes \Pi_x$ Choi matrix support projection, simplifies SDP $\Upsilon(K)$...

$$\Upsilon(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes \mathbb{1},$$
$$\operatorname{Tr}_A U = \mathbb{1}^B,$$
$$\Pi(S \otimes \mathbb{1} - U) = 0.$$

[R. Duan / AW, arXiv:1409.3426]

$$\Upsilon(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x^\perp,$$

$$\sum_x (R_x + s_x \Pi_x) = \mathbb{1}.$$

[R. Duan/AW, arXiv:1409.3426]

$$\Upsilon(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x^\perp,$$

$$\sum_x (R_x + s_x \Pi_x) = \mathbb{1}.$$

$$\leq A(K) := \max \sum_x s_x \text{ s.t. } 0 \leq s_x,$$

$$\sum_x s_x \Pi_x \leq \mathbb{1}.$$

[R. Duan/AW, arXiv:1409.3426]

$$\Upsilon(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x^\perp,$$

$$\sum_x (R_x + s_x \Pi_x) = \mathbb{1}.$$

$$\leq A(K) := \max \sum_x s_x \text{ s.t. } 0 \leq s_x,$$

$$\sum_x s_x \Pi_x \leq \mathbb{1}.$$

Semidefinite packing number; also reduces to fractional packing no. in classical case, but is multiplicative: $A(K \otimes K') = A(K)A(K')$.

[R. Duan/AW, arXiv:1409.3426]

$$\Upsilon(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x^{\perp},$$

$$\sum_x (R_x + s_x \Pi_x) = \mathbb{1}.$$

$$\leq A(K) := \max \sum_x s_x \text{ s.t. } 0 \leq s_x,$$

$$\sum_x s_x \Pi_x \leq \mathbb{1}.$$

Thm. $C_{ONS}(K) = \lim \frac{1}{n} \log \Upsilon(K^{\otimes n}) = \log A(K).$

[R. Duan/AW, arXiv:1409.3426]

$$\Upsilon(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x^\perp,$$

$$\sum_x (R_x + s_x \Pi_x) = \mathbb{1}.$$

$$\leq A(K) := \max \sum_x s_x \text{ s.t. } 0 \leq s_x,$$

$$\sum_x s_x \Pi_x \leq \mathbb{1}.$$

Thm. $C_{ONS}(K) = \lim \frac{1}{n} \log \Upsilon(K^{\otimes n}) = \log A(K).$

Show actually $\Upsilon(K^{\otimes n}) \geq n^{-c} A(K)^n$, starting from an optimal solution for $A(K)$; then by group (permutation) symmetry that we can satisfy the extra constraints loosing little..

$$\Upsilon(K) = \max \sum_x s_x \text{ s.t. } 0 \leq R_x \leq s_x \Pi_x^\perp,$$
$$\sum_x (R_x + s_x \Pi_x) = \mathbb{1}.$$

$$\leq A(K) := \max \sum_x s_x \text{ s.t. } 0 \leq s_x,$$
$$\sum_x s_x \Pi_x \leq \mathbb{1}.$$

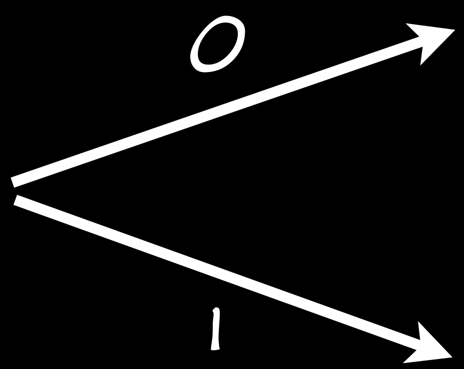
Thm. $C_{ONS}(K) = \lim \frac{1}{n} \log \Upsilon(K^{\otimes n}) = \log A(K).$

Thm. $G_{ONS}(K) = \log \Sigma(K)$ asympt. simul. cost

$$\Sigma(K) = \min \operatorname{Tr} T \text{ s.t. } 0 \leq V_x \leq T,$$
$$\operatorname{Tr} V_x = 1,\ V_x \leq \Pi_x.$$

[R. Duan/AW, arXiv:1409.3426]

Example: Two-pure-state cq-channel



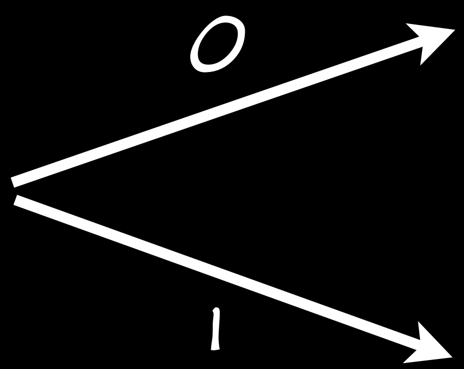
$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi_1\rangle = \alpha|0\rangle - \beta|1\rangle$

$K = \mathrm{span}\{|\psi_0\rangle\langle 0|, |\psi_1\rangle\langle 1|\}$

$(1 > \alpha > \beta > 0;\ \alpha^2 + \beta^2 = 1)$

Example: Two-pure-state cq-channel



$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$

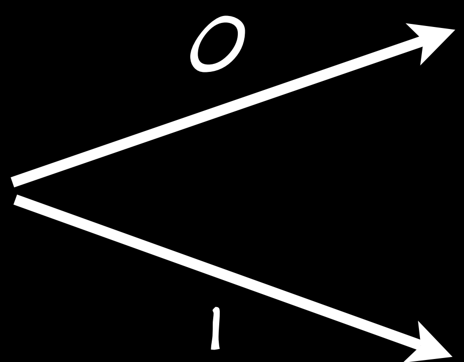
$|\psi_1\rangle = \alpha|0\rangle - \beta|1\rangle$

$K = \mathrm{span}\{|\psi_0\rangle\langle 0|,\ |\psi_1\rangle\langle 1|\}$

$(1 > \alpha > \beta > 0;\ \alpha^2 + \beta^2 = 1)$

$\Upsilon(K) = 1$, but $\Upsilon(K \otimes K) \geq 1/(\alpha^4 + \beta^4)$, and for n large enough, $\Upsilon(K^{\otimes n}) \geq 1/(\alpha^{2n} + \beta^{2n})$.

Example: Two-pure-state cq-channel



$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi_1\rangle = \alpha|0\rangle - \beta|1\rangle$

$K = \mathrm{span}\{|\psi_0\rangle\langle 0|, |\psi_1\rangle\langle 1|\}$

$(1 > \alpha > \beta > 0;\ \alpha^2 + \beta^2 = 1)$

$\Upsilon(K) = 1$, but $\Upsilon(K \otimes K) \geq 1/(\alpha^4 + \beta^4)$, and for n large enough, $\Upsilon(K^{\otimes n}) \geq 1/(\alpha^{2n} + \beta^{2n})$.

Easy: $A(K) = 1/\alpha^2$; $\Sigma(K) = 1 + 2\alpha\beta$.

5. Lovász number encore

Now the best: Minimize $\mathbf{A}(K)$ over all cq-channels with the same confusability graph G ($x \sim x'$ iff $\Pi_x \not\perp \Pi_{x'}$).

5. Lovász number encore

Now the best: Minimize $A(K)$ over all cq-channels with the same confusability graph G ($x \sim x'$ iff $\Pi_x \not\perp \Pi_{x'}$).

Thm. $\min A(K) = \vartheta(G)$; $\min C_{0NS}(K) = \log \vartheta(G)$.

In words: Lovász' number gives the no-signalling assisted capacity of the worst cq-channel with confusability graph G.

First capacity interpretation of $\vartheta(G)$:-)

[R. Duan/AW, arXiv:1409.3426]

6. Last words:

- SDP formulas for assisted capacity and simulation cost (one-shot)
- SDP can regularize to a relaxed SDP :-)
- Capacity interpretation of Lovász number
- Gap between $C_{0E}(G)$ and $\log \vartheta(G)$?
- Regularization necessary? There could be K such that $\Upsilon(K) = \vartheta(G)$ – cf. Ching-Yi Lai's poster on Monday!
- $\Sigma(G) := \min \{ \Sigma(K) : G \supset K^\dagger K \} = ??$ Know only: between $\vartheta(G)$ and $\alpha^*(G)$