local tests of global entanglement
and a counterexample to the generalized area law
1. **q. expanders**
   maximally entangled states

2. **entanglement**
   testing and communication

3. **area law**
   gaps, connections, correlations
Quantum Expanders
1 Classical expanders

- walk on these graphs? mix fast!
Classical expanders

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- divide in two? cut a lot (fraction) of edges!
Classical expanders

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- divide in two? cut a lot (fraction) of edges!

examples: Ramanujan, Cayley
1 Classical expanders

- walk on these graphs? mix fast!
- divide in two? cut a lot (fraction) of edges!
- examples: Ramanujan, Cayley

- explicit, constant-degree approximations to the full graph

- normalized adjacency matrix
- second largest eigenvalue $1-\lambda$

- a review [Hoory Linial Wigderson]
- a talk [Harrow quantum expanders youtube]
Mixing up something quantum

- applying random unitaries

\[ \mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger \]

- classical expanders: explicit, constant-degree approximations to the full graph fast-mixing
1 Quantum expanders

- applying random unitaries from a small set a discrete approx. to Haar

\[ \mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger \]

- classical expanders: explicit, constant-degree approximations to the full graph fast-mixing
1 Quantum expanders

- applying random unitaries from a small set a discrete approx. to Haar

- transform $N \times N$ matrices $\mathbb{I}$ stays, everything else changes

$$\|\mathcal{E}(X)\|_2 \leq \lambda \|X\|_2$$

small 2\textsuperscript{nd} largest sing. value

- QE constructions for fixed $k$ (... 8, 4, 3)

$1 - \lambda \approx k^{-c}$

[Ben-Arroyo+ 07, Hastings ’07, Gross & Eisert ’08, Hast. & Harr. ’09, Gross ‘15]
Quantum expanders

- Transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger$$

A matrix that doesn't change?

$$X = I$$
1 Quantum expanders

- transform $N \times N$ matrices

\[
\mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger
\]

a matrix that doesn't change?

\[X = \mathbb{I}\]

\[U_i^\dagger \]

\[U_i \]

\[X \]

\[U_i \]

\[U_i^\dagger \]

\[U_i X = X U_i\]
1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger$$

a matrix that doesn’t change?

$$X = \mathbb{I}$$

- interpreting matrices as 2-register states

$$X = \sum_{a, b} X_{ab} |a\rangle \langle b|$$
1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger$$

- a matrix that doesn’t change?

$$X = \mathbb{I}$$

- interpreting matrices as 2-register states

$$|X\rangle = \sum_{a,b} X_{ab} |a\rangle |b\rangle$$

pure state
1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger$$

a matrix that doesn’t change?

$$X = \mathbb{I}$$

- interpreting matrices as 2-register states

$$\frac{1}{k} \sum_{i=1}^{k} (U_i \otimes U_i^*) \sum_{a,b} X_{ab} |a\rangle |b\rangle$$

applying an expander distributively

a stationary state? max. entangled!

$$\frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle |x\rangle$$
1 Quantum expanders & 2 registers

- transform $N \otimes N$ states
  close to the depolarizing channel

\[ \| \tilde{\mathcal{E}} - |\phi_N\rangle \langle \phi_N | \| = \lambda \]

- interpreting matrices as 2-register states

\[ \frac{1}{k} \sum_{i=1}^{k} (U_i \otimes U_i^*) \sum_{a,b} X_{ab} |a\rangle |b\rangle \]

applying an expander distributively

a stationary state? max. entangled!

\[ \frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle |x\rangle \]
local tests of global entanglement
EPR testing

- how costly is it to certify that we share a maximally entangled state?

\[
\frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle |x\rangle
\]

\(O(\log \log(N) + \log(1/\lambda))\) qubits. [BDSW ’96, BCGST ’02]
2 EPR testing

how costly is it to certify that we share a maximally entangled state?

\[ \frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle |x\rangle \]

\( O(\log(1/\lambda)) \) qubits for error \( \lambda \).
2 EPR testing

- how costly is it to certify that we share a maximally entangled state?
- apply a quantum expander distributively

\[
\frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle|x\rangle
\]

\[
U_i \otimes U_i^*
\]
EPR testing

- when does the qutrit remain uniform?

\[
\frac{1}{\sqrt{3}} \sum_{i=1}^{3} |i\rangle (U_i \otimes U_i^*) \sum_{a,b} X_{ab} |a\rangle |b\rangle
\]
2 EPR testing

- when does the qutrit remain uniform? for max. entangled $X$

- quantum expander property … soundness

\[
\frac{1}{\sqrt{3}} \sum_{i=1}^{3} |i\rangle (U_i \otimes U_i^*) \frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle |x\rangle
\]

\[
\text{constant message length} \quad \text{constant } p_{\text{fool}} : \text{uniform?}
\]
little communication
local tests
constant error

global correlations
a counterexample to the generalized area law
few connections
local interactions
constant gap

local correlations
few connections
local interactions
constant gap
global correlations
3. Ground states of gapped quantum spin systems

- entanglement entropy

\[ S = -\text{Tr}(\rho_A \ln \rho_A) \sim \text{volume} \]

\[ \rho_A = \text{Tr}_B \rho \]

area law → a simple ground state?
Gapped 1D Hamiltonians

- Nothing closer than $\Delta$ to the ground state.

the AKLT (spin-1) chain

$$
\sum_{j=1}^{N-1} \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} \left( \vec{S}_j \cdot \vec{S}_{j+1} \right)^2
$$

a biased walk in 1D

$$
\sum_{j=1}^{N-1} (|j\rangle - B|j+1\rangle) (\langle j| - B\langle j+1|)
$$

exp. falloff of correlations, MPS/DMRG work well
Without a gap, the entropy can be large. [Verstraete, Latorre+]}
3. Ground states of gapped H's & the area law

- entanglement entropy

\[ S = -\text{Tr}(\rho_A \ln \rho_A) \sim \text{volume} \]

surface area

a gapped system → an area law
3. Ground states of gapped H’s & the area law

- entanglement entropy

\[ S = -\text{Tr}(\rho_A \ln \rho_A) \sim \text{volume} \]

\text{surface area}

1D ... theorems [Hastings 07, Arad+ 13]

algorithms [White 92, Vidal 03, Landau+ 13]

2D ... we’re close

a gapped system  →  an area law
3. Ground states of gapped H’s & the area law

- entanglement entropy
  \[ S = -\text{Tr}(\rho_A \ln \rho_A) \sim \text{volume} \text{ surface area} \]

1D ... theorems [Hastings 07, Arad+ 13]

algorithms [White 92, Vidal 03, Landau+ 13]

2D ... we’re close

- generalized area conjecture entropy \sim \text{cut size}
a gap
a few links
O(1) terms

not much entanglement
(a “simple” ground state)

generalized area conjecture
entropy ~ cut size
3 Generalized area conjecture: the counterexample

- an $N \times 3 \times 3 \times N$ dimensional system
- a gapped, frustration-free Hamiltonian
  an $O(1)$ interaction between the qudits
3 Generalized area conjecture: the counterexample

- an $N \times 3 \times 3 \times N$ dimensional system
- a gapped, frustration-free Hamiltonian
  an $O(1)$ interaction between the qudits
- a unique, very entangled ground state
  $O(N)$ entanglement entropy across the cut
The 4-particle \((N \times 3 \times 3 \times N)\) Hamiltonian

- a projector \(P_L\) with ground states

\[
\frac{1}{\sqrt{3}} \left( |1\rangle |x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle \right)
\]

- as a vector

\[
\begin{align*}
  x & \\
  Ax & \otimes |j\rangle \otimes |y\rangle \\
  Bx &
\end{align*}
\]

- as a matrix

\[
\begin{align*}
  X_1 & \\
  AX_1 & \\
  BX_1 & \\
  X_2 & \\
  AX_2 & \\
  BX_2 & \\
  X_3 & \\
  AX_3 & \\
  BX_3 &
\end{align*}
\]
The 4-particle \((N \times 3 \times 3 \times N)\) Hamiltonian

- a projector \(P_R\) with ground states

\[ \frac{1}{\sqrt{3}} (|1\rangle y + |2\rangle A y + |3\rangle B y) \]

- as a vector

\[ |i\rangle \otimes |x\rangle \otimes \begin{array}{ccc} y & yA & yB \\ Y_1 & Y_1A & Y_1B \\ Y_2 & Y_2A & Y_2B \\ Y_3 & Y_3A & Y_3B \end{array} \]

- as a matrix

\(P_R\)
3. The 4-particle $(N \times 3 \times 3 \times N)$ Hamiltonian

- a projector $P_L$
  
- a projector $P_R$

\[ (|1\rangle|x\rangle + |2\rangle|A|x\rangle + |3\rangle|B|x\rangle) / \sqrt{3} \]

\[ (|1\rangle|y\rangle + |2\rangle|A|y\rangle + |3\rangle|B|y\rangle) / \sqrt{3} \]
The 4-particle \((N \times 3 \times 3 \times N)\) Hamiltonian

- a projector \(P_L\)
- a projector \(P_R\)
- a projector \(P_M\)

\[
\frac{\langle 1|x \rangle + \langle 2|A|x \rangle + \langle 3|B|x \rangle}{\sqrt{3}}
\]

\[
\frac{\langle 1|y \rangle + \langle 2|A|y \rangle + \langle 3|B|y \rangle}{\sqrt{3}}
\]
The 4-particle \((N \times 3 \times 3 \times N)\) Hamiltonian

- a projector \(P_L\)
- a projector \(P_R\)
- a projector \(P_M\)

\[
\frac{\left( |1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle \right)}{\sqrt{3}}
\]

\[
\frac{\left( |1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle \right)}{\sqrt{3}}
\]

enforce symmetry: 12 & 21
13 & 31

who commutes with \(A\) and \(B\)?

only the identity, as \([I, A, B]\) is a q. expander

\[
\begin{align*}
X & & \textcolor{orange}{XA} & & \textcolor{red}{XB} \\
\textcolor{grey}{AX} & & \textcolor{orange}{AXA} & & \textcolor{grey}{AXB} \\
\textcolor{red}{BX} & & \textcolor{grey}{BXA} & & \textcolor{grey}{BXB}
\end{align*}
\]
The 4-particle \((N \times 3 \times 3 \times N)\) Hamiltonian

- a projector \(P_L\)
- a projector \(P_R\)
- a projector \(P_M\)

\[
\begin{align*}
(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle) / \sqrt{3} \\
(|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle) / \sqrt{3}
\end{align*}
\]

enforce symmetry: 12 & 21
13 & 31

who commutes with \(A\) and \(B\)?

only the identity, as \([I, A, B]\) is

a q. expander

\[
\frac{1}{3\sqrt{N}}
\]
The 4-particle \((N \times 3 \times 3 \times N)\) Hamiltonian

a unique, very entangled ground state

a gapped Hamiltonian, \(O(1)\) terms, frust. free

\[
\frac{1}{3\sqrt{N}}
\]
Making the counterexample local

- a quantum circuit, history state
  Kitaev’s LH, 1D, qudits

\[
\begin{align*}
\text{prepare} & \quad \frac{1}{\sqrt{3}} (|1\rangle x + |2\rangle A|x\rangle + |3\rangle B|x\rangle) \\
\text{from} & \quad \frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |3\rangle) |x\rangle
\end{align*}
\]
3 Making the counterexample local

- a quantum circuit, history state
  Kitaev’s LH, 1D, qudits

\[
\text{prepare } \frac{1}{\sqrt{3}} (|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle)
\]

\[
\text{from } \frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |3\rangle) |x\rangle
\]
3 Making the counterexample local

- a quantum circuit, history state
  Kitaev’s LH, 1D, qudits

\[ H_K \rightarrow 0 \]

\[ |\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle \]

\[ \underbrace{U_t \cdots U_1|\varphi_0\rangle}_{0} \]
The history state: a ground state of a qudit chain

2-local conditions

\[ |\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle \]

\(d=13\) \[\text{AGIK’08}\]
\(d=8\) \[\text{HNN’13}\]
The history state: a ground state of a qudit chain

2-local conditions

Clock encoding
State progression
Initialization

\[ \cdots |0\rangle \otimes |0\rangle \]

\[ |\varphi_t\rangle \otimes |t\rangle \]

\[ |\varphi_{t+1}\rangle \otimes |t + 1\rangle \]

\[ |\psi_{\text{hist}}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle \]

Most of the state has the result
Making the counterexample local

- a quantum circuit, history state
  Kitaev’s LH, 1D, qudits
  an approx. groundstate, a small $1/\text{poly}(n)$ gap
Making the counterexample local

- a quantum circuit, history state
  Kitaev’s LH, 1D, qudits
  an approx. groundstate, a small $1/\text{poly}(n)$ gap

- rescale $P_L$, $P_R$ (not the middle!)
  a constant gap, huge couplings
Making the counterexample local

- a quantum circuit, history state
  Kitaev’s LH, 1D, qudits
  an approx. groundstate, a small $1/\text{poly}(n)$ gap

- rescale $P_L$, $P_R$ (not the middle!)
  a constant gap, huge couplings

- decompose using gadgets [Cao, N.]
  huge couplings many spins, high degree
3 A local Hamiltonian, \((N \times M) \times 3 \times 3 \times (N \times M)\)

- a unique and very entangled ground state

\[ \approx |w\rangle \otimes \]

- a constant gap

O(1) or smaller norm terms, frustration
What next?

- a longer middle chain?
What next?

- a longer middle chain?
- a nice lattice on the sides?
3 What next?

- a longer middle chain?
- a nice lattice on the sides?
- a new area conjecture: count the cut & things nearby?
1. q. expanders
   maximally entangled states

2. entanglement
   testing and communication

3. area law
   gaps, connections, correlations
local tests of global entanglement
and a counterexample to the generalized area law