Local Operations in Fault-tolerant Quantum Computation: Gauge Color Codes

Héctor Bombín
outline

• introduction
• locality in fault-tolerant quantum comp.
• topological codes & local operations
• results
• single-shot error correction
• self-correction
• gauge color codes
• universality via gauge-fixing
error correction

For quantum computation...

• want: isolation + control
• have: decoherence + imprecision
• need: error correction
• how: one qubit encoded in many physical qubits

logical qubit
error correction

- extra degrees of freedom detect errors
- check operators fix the code subspace
- measuring them gives the error syndrome
- to correct, guess error from syndrome

![Diagram](image-url)
locality

- correction is possible if errors are not arbitrary
- local errors are more likely
- phenomenology: local stochastic noise

\[ P(\text{error affects qubits } i_1, i_2, \ldots, i_n) \leq \varepsilon^n \]

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○

more likely  less likely
fault-tolerant QC

- compute with encoded qubits
- errors pile up, but error correction flushes them away (up to a point)
- logical operations should preserve locality!
transversal operations

- act separately on physical subsystems
- do not spread errors
- downside: never universal  Eastin & Knill ‘09
transversal operations

• act separately on physical subsystems
• do not spread errors
• downside: never universal  Eastin & Knill ‘09
local operations

- finite depth circuit
- limited spread of errors
- in some contexts, limited power

Bravyi & König ’09,...
local operations

- finite depth circuit
- limited spread of errors
- in some contexts, limited power \textsuperscript{Bravyi & König '09,...}
quantum-local operations

• finite depth circuit + global classical comp.
• universal operations + error correction **no limits!**
quantum-local operations

- finite depth circuit + global classical comp.
- universal operations + error correction

Caution!

noiseless classical comp.  

no limits!
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topological codes
Kitaev ‘97

- physical qubits on a lattice
- local check operators
- ‘local’ operators cannot harm logical qubits
topological codes

Kitaev '97

low $\varepsilon$  error threshold /  phase transition  high $\varepsilon$

perfect correction  ↔  large systems  →  information destroyed
topological order

- gapped (local) quantum Hamiltonian
- locally undistinguishable ground states
- robust against deformations

\[ H = -J \sum_i P_i \]
self-correction

- for $D \geq 4$ excitations can be extended objects

\begin{align*}
\text{low } T & \quad \text{confined} \\
T_C & \\
\text{high } T & \quad \text{unconfined}
\end{align*}

perfect protection $\leftrightarrow$ large systems $\leftrightarrow$ information destroyed

Dennis et al '02
local operations

- geometrically local, finite depth circuit
- finite spatial spread of errors
Dimensional restrictions

- top. stabilizer codes: check ops in Pauli group
- geometrical constraints on local gates

\[ \mathcal{P}_D := \{ U \mid U \mathcal{P} U^\dagger \subseteq \mathcal{P}_{D-1} \}, \quad \mathcal{P}_1 := \mathcal{P} \]

\[ R_D := \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^D} \end{pmatrix} \]
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**color codes**

- topological stabilizer codes defined for any D
- optimal transversal gates: $R_D$ transversal
subsystem codes

- gauge (free) degrees of freedom
- in topological codes, can be local
- more local measurements
- **gauge fixing**: gauge ops $\rightarrow$ check ops
- amounts to error correction
- allows to combine properties of codes (e.g. transversal gates for universality)

Poulin '05

Paetznick & Reichardt '13
3D gauge color codes

- 6-local measurements, as in 2D
- universal transversal gates via gauge fixing

CNOT + H
CNOT + T

gauge
conventional

arXiv:1311.0879
3D gauge color codes

- dimensional jumps via gauge fixing
- 2D color codes require much less qubits
quantum-local error correction

- in topological stabilizer codes **ideal** error correction is q-local
- but real measurements are **noisy**, and multiple rounds are required (to avoid large errors)
quantum-local error correction

- some codes are inherently robust!
- local measurement errors yield local errors
- **single-shot error correction** (no multiple rounds)
- linked to self-correction: confinement

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**syndrome extraction**
- local, quantum

**decoding**
- global, classical

**correction**
- transversal, quantum

arXiv:1404.5504
quantum-local error correction

- 3D gauge color codes are single-shot!
- confinement due to gauge ‘redundancy’
- also single-shot gauge-fixing

syndrome extraction
local, quantum

decoding
global, classical

correction
transversal, quantum

arXiv:1404.5504
3D-local constant time QC

- fault-tolerant QC in 3D qubit lattice
- local quantum ops + global classical comp.
- constant time ops. (disregarding efficient CC)

FIG. 1: A 3D layout for fault-tolerant quantum computing.
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lsing model

- simplest (classical) self-correction
- critical temperature $T_C$ if $D>1$
- below $T_C \rightarrow \text{confined loops}$
- stable bit (exponential lifetime)
repetition code à la Ising

- stabilizer code for bit-flip errors
- qubits = faces
- check operators = edges

\[ Z_e := Z_i Z_j \]

- syndrome composed of loops
- low local noise \(\rightarrow\) confined loops
noisy error correction

- assume noisy measurements only
- goal: confined residual loops
noisy error correction

effective wrong measurements = residual syndrome
spatial dimension

- 1D Ising / repetition code: \textbf{unconfined} excitations / syndrome

- confinement mechanism: extended excitations

- full quantum self-correction seems to require $D>3$

Dennis et al '02
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confinement in 3D

- 3D gauge color codes:
- errors: string-net like
- syndrome: endpoints
- conserved color charge
- direct measurement of syndrome: no confinement
- instead, obtain it from gauge syndrome
- another application of subsystem codes!
but, instead, it is possible to perform a collection of local extraction of the error syndrome. This provides no confinement points of these strings, all living in a 3D lattice. Direct confinement can be visualized as strings and syndromes as the end-gauge color codes. As in the 1D repetition code, errors can be corrected at the level of noisy correction. In particular, again one finds that confinement results in single-shot error-correction. This translates nicely when moving to active error correction. Involving that all excitations should be extended objects. Again the mechanism for confinement is the spatial dimension needs to be at least four, and thus there is no confinement. This is possible if and only if an odd number of ry-edges, or of yg-edges, meet at a b-vertex, then it is an error syndrome vertex (and then an odd number of gauge syndrome rg-edges meet at b). A vertex is in the gauge syndrome edge collection because a vertex has an odd number of gauge syndrome edges of any color incident on it. The error syndrome consists of a collection of vertices. The gauge syndrome consists of a collection of edges that has a given color on them has to be localized and has to be applied separately to each of them. If these clusters are confined in the sense discussed. Indeed, the edges of the gauge syndrome loops are confined. Thus indeed residual syndrome loops are confined. Fortunately, for the constraints imposed by the relationship to the original error syndrome, we have that logical errors are unlikely when both the corresponding 1 punctual and can move freely without any energy cost: for such events decreases exponentially with the lattice size, and the probability of measurements will only happen when the clusters are large compared to the system size, and the probability that a set of edges is a subset of the lattice, the set is composed of loops. In other words, the gauge syndrome: endpoints = syndrome of faults. The situation is somewhat similar when quantum self-correction. For D = 1 spatial dimensions the Ising model fails to be understood. However, for known self-correction, the Ising model can be applied separately to each of them. If these clusters are confined in the sense discussed. Indeed, the edges of the gauge syndrome loops are confined. Thus indeed residual syndrome loops are confined. An interesting way to choose the correcting set is to correct each connected component of the gauge syndrome. Here we consider the gauge syndrome edges of any color: red, green, blue, and yellow (r, g, b, y).

Details are as follows. The lattice has vertices with four different colors: red, green, blue, and yellow (r, g, b, y). Each vertex is linked by the edge, which has to be different. E.g., a rg-edge connects a b-vertex and a y-vertex. The vertices linked by the edge, which have to be different colors, are those complementary to the colors of the two vertices linked by the edge. The position of the error syndrome points is linked to the different colors of the two vertices linked by the edge. Namely those complementary to the colors of the two vertices linked by the edge. The resulting confinement in 3D is composed of loops. In other words, the gauge syndrome: endpoints = syndrome of faults.
confinement in 3D

- the gauge syndrome is unconfined, it is random except for the fixed branching points.
- the (effective) wrong part of the gauge syndrome is confined.
- each connected component has branching points with neutral charge (i.e. locally correctable).
- branching points exhibit charge confinement!
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gauge fixing

- there is an X and a Z gauge syndrome
- any of them can be fixed to become part of the stabilizer, but not both!
- each option corresponds to a conventional 3D color code

\[ \text{transversal} \quad \xrightarrow{H} \quad \text{self-dual} \quad \xleftarrow{\text{fixed } X} \quad \text{transversal} \quad \xrightarrow{\text{transversal } T} \quad \text{fixed } Z \quad \xleftarrow{\text{transversal } HTH} \quad \text{fixed } X \]
### gauge fixing

<table>
<thead>
<tr>
<th>Syndrome geometry</th>
<th>fixed Z</th>
<th>fixed X</th>
<th>self-dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>X check ops</td>
<td>![Green Circle]</td>
<td>![Red and Blue Circle]</td>
<td>![Green Circle]</td>
</tr>
<tr>
<td>Z check ops</td>
<td>![Red and Blue Circle]</td>
<td>![Green Circle]</td>
<td>![Green Circle]</td>
</tr>
</tbody>
</table>

**TQFT:**
- Homological
- **?**
summary & future work

- color codes have optimal transversal gates
- universality via gauge fixing
- single-shot error correction is possible and is linked to self-correction
- 3D-local FTQC with constant time overhead
- what are the limitations in 2D?
- what about non-geometrical locality?
- related 3D self-correcting systems?
New QI group in COPENHAGEN!

Wanted: Phd students & postdocs
Masterclass on quantum mathematics
May 18-22 2015

Michael Freedman, Bruno Nachtergaele, Robert Seiringer, Spiros Michalakis, and more

http://www.math.ku.dk/english/research/conferences/2015/qmath-masterclass/