On the informational completeness of local observables

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Motivation

Curse of dimensionality: For problems that involve many degrees of freedom, the dimension of the phase space blows up exponentially.

- Dimension of the quantum state that describes a $n$-particle system grows as exponentially in $n$. This can be problematic for many tasks, such as:
  - Performing quantum state tomography
  - Performing quantum state verification
  - Studying many-body Hamiltonian

Goal: find a large class of states $S$ such that:

- Some of these tasks can be done efficiently.
- If a state is in $S$, one can efficiently verify that fact.
- The above features remain robust against imperfect measurements/finite precision.
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- Some of these tasks can be done efficiently.
- If a state is in $S$, one can efficiently verify that fact.
- The above features remain robust against imperfect measurements/finite precision.
Brief summary

I will propose a class of states over $n$ particles, $S_n$ that has the following features.

- One can verify that the state is in $S_n$ with $O(n)$ measurement/computation time.
- Any state in $S_n$ is defined by a set of $O(1)$-particle density matrices.
  - State tomography/verification can be done with $O(n)$ measurement/computation time.
  - Small errors in the $O(1)$-particle density matrices don’t propagate too much. (robust error bound)

The class includes highly entangled states (e.g., topological code, quantum Hall system).
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  - State tomography/verification can be done with \( O(n) \) measurement/computation time.
  - Small errors in the \( O(1) \)-particle density matrices don’t propagate too much (robust error bound).
- The class includes highly entangled states (e.g., topological code, quantum Hall system).
Informational completeness

Setup: Suppose we are given a quantum state $\rho$ describing $n$ qubits. We know some of its expectation values.

1. $\text{Tr}(\rho) = 1.$
2. $\rho \geq 0.$
3. $\text{Tr}(\rho \sigma_i) = \langle \sigma_i \rangle$, $i \in I$.

$I$: some finite set.

Specifying $\rho$: Assign expectation values for all linearly independent observables. (≈ $4^n$)

Such observables are informationally complete: their expectation values completely determine the state.
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What if we do not specify all the expectation values of the linearly independent observables? : The problem is inherently ill-defined. **Or, is it?**
Sometimes, expectation values of local observables completely determine the global state.
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1. **Product state**: \( |\psi\rangle = |0\rangle \otimes |1\rangle \otimes \cdots \otimes |1\rangle \).

2. **Matrix product states**: \( \sum_{s_1, \ldots, s_n} \text{Tr}(A^{s_1} \cdots A^{s_n}) |s_1\rangle \otimes \cdots \otimes |s_n\rangle \)

[Cramer et al. 2011]
Matrix product states

For (injective) matrix product states, local observables can be informationally complete.

\[ \sum_{s_1, \ldots, s_n} \text{Tr}(A^{s_1} \cdots A^{s_n}) \ket{s_1} \otimes \cdots \otimes \ket{s_n} \]
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\( \rho_{34} \)
Matrix product states

For (injective) matrix product states, local observables can be informationally complete.

\[ \rho_{12}, \rho_{23}, \cdots \rightarrow \text{MPS tomography algorithm} \rightarrow \text{Output} \]

Output: MPS $|\psi'\rangle$ that is consistent with $\rho_{12}, \rho_{23}, \cdots$ with a certificate showing that $|\langle \psi' | \psi_{\text{real}} \rangle| \geq 1 - \epsilon$. [Cramer et al. 2011]
Takeaway message

Given a set of expectation values of local observables, there exists an efficiently checkable condition that tells you whether they are informationally complete.
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Our result can be thought as a generalization of the result of Cramer et al. to higher dimensional systems, but with an important difference.

- Cramer et al. appeals to the special structure of the MPS, but our approach does not involve any global wavefunction at all.
Setup
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For all sites $k$, we know the reduced density matrices over the neighborhood of $k$.

- $k$ : Site
- $\mathcal{N}_k$ : Neighborhood of $k$.

Question: If one can find a state $\rho'$ that is consistent with $\{ \rho_{k\mathcal{N}_k} \}$, is it close to $\rho$?
Main result (Colloquial version)

There exists a certificate $\epsilon(\{\rho_k\mathcal{N}_k\})$ such that,

$$|\rho - \rho'|_1 \leq \epsilon(\{\rho_k\mathcal{N}_k\}).$$

- Efficiency: $O(n)$ measurement/computation time.
- Applicability: any 1D/2D gapped system assuming a certain form of area law holds, but possibly more.
  - Both with and without topological order!
- Robustness: if $|\rho_k\mathcal{N}_k - \rho'_k\mathcal{N}_k| = \epsilon$, there is an additional error term which is $O(n\epsilon \log \frac{1}{\epsilon})$. 
Applications

- Quantum state tomography
- Quantum state verification
- Possibly more?
The plan

Recall our main result:

\[ |\rho - \rho'|_1 \leq \epsilon(\{\rho_{kN_k}\}) \]

1. **Globally Computable Upper Bound**: I will upper bound a trace distance between \( \rho \) and \( \rho' \) by a quantity that can be computed from the global states.

2. **Locally Computable Upper Bound**: Using information inequalities, I will introduce a new quantity that can be computed from local reduced density matrices. This quantity will be an upper bound of the GCUB.

3. I will show that the LCUB is small for many systems.
Globally Computable Upper Bound: background

\[ I(A : C|B) = S(AB) + S(BC) - S(B) - S(ABC) \] is quantum conditional mutual information.

\[ I(A : C|B) \geq 0. \]

[Lieb, Ruskai 1972]

\[ *S(A) = -\text{Tr}(\rho_A \log \rho_A) \] : entanglement entropy of A.
Globally Computable Upper Bound: background

[Petz, 1984]

\[ I(A : C|B) = 0 \]

if and only if

\[ \rho_{ABC} = \rho_{BC}^{\frac{1}{2}} \rho_{B}^{-\frac{1}{2}} \rho_{AB}^{\frac{1}{2}} \rho_{B}^{-\frac{1}{2}} \rho_{BC}^{\frac{1}{2}}. \]

* The precise form of the equation does not matter for the purpose of this talk. What matters is the fact that the global state is completely determined by its local reduced density matrices, if the conditional mutual information is 0.
Suppose $\rho_{ABC}$ and $\sigma_{ABC}$ are locally consistent, i.e.,

$$\rho_{AB} = \sigma_{AB}, \quad \rho_{BC} = \sigma_{BC},$$

and $I(A : C|B)_\rho = I(A : C|B)_\sigma = 0$.

$$\rho_{ABC} = \sigma_{ABC} = \rho_{BC}^{1/2} \rho_{B}^{-1/2} \rho_{AB} \rho_{B}^{-1/2} \rho_{BC}^{1/2}.$$

If $\rho_{ABC}$ and $\sigma_{ABC}$ are conditionally independent, and their marginal distributions over $AB$ and $BC$ are consistent, they are globally equivalent.
Globally Computable Upper Bound

Colloquially: If $\rho_{ABC}$ and $\sigma_{ABC}$ are approximately conditionally independent, i.e.,

$$I(A : C | B)_\rho \approx 0, \quad I(A : C | B)_\sigma \approx 0$$

and $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\rho_{ABC} \approx \sigma_{ABC}.$$
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and $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\rho_{ABC} \approx \sigma_{ABC}.$$

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|^2_1 \leq \frac{1}{2} (I(A : C|B)_{\rho} + I(A : C|B)_{\sigma}).$$

* If $\rho_{AB} \approx \sigma_{AB}$ and $\rho_{BC} \approx \sigma_{BC}$, there is an additional additive contribution proportional to $\log(\text{dimension})$. 

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Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A : C|B)_\rho + I(A : C|B)_\sigma).$$

$\rho_{ABC}$ and $\sigma_{ABC}$ are close to each other if

- Their marginal distribution over $AB$ and $BC$ are the same, and
- $I(A : C|B)_\rho \approx 0$ and $I(A : C|B)_\sigma$. 
Locally Computable Upper Bound

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2} (I(A:C|B)_{\rho} + I(A:C|B)_{\sigma}).$$

Okay that is kind of cool. I guess you are trying to use this result to bound the trace distance between two states from its local reduced density matrices?
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A : C|B)_\rho + I(A : C|B)_\sigma).$$

But that is never going to work. You see, in order to compute the upper bound, you need to know the entropy of the global states.
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|^2_1 \leq \frac{1}{2} (I(A : C | B)_\rho + I(A : C | B)_\sigma).$$

I mean, let’s suppose, WLOG, A is a very large region like this.
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2} (I(A : C|B)_\rho + I(A : C|B)_\sigma).$$

... and B and C are chosen like this.
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} \left| \rho_{ABC} - \sigma_{ABC} \right|^2 \leq \frac{1}{2} \left( I(A : C|B)_\rho + I(A : C|B)_\sigma \right).$$

Remember the setup?

For all sites $k$, we know the reduced density matrices over the neighborhood of $k$.
- $k$: Site
- $\mathcal{N}_k$: Neighborhood of $k$

If one can find a state $\rho'$ that is consistent with $\{\rho_{kN_k}\}$, is it close to $\rho$?
Locally Computable Upper Bound

How do you propose to compute $I(A:C|B)$, knowing only the density matrices over each sites and its neighbours?
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$, then

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|^2_1 \leq \frac{1}{2} (I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

Recall $I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC)$.
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2} (I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

S(BC) and S(B) can be computed easily from the given local density matrices.

$$I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC)$$
The nontrivial part is the remaining term, $S(AB) - S(ABC)$.

\[ I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC) \]
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$, then

$$I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC)$$

I will give you an upper bound on $S(AB) - S(ABC)$ which can be computed from the given reduced density matrices.
Strong subadditivity asserts that

\[ S(AB) + S(BC) - S(B) - S(ABC) \geq 0 \]

for any tripartite state \( \rho_{ABC} \).

Weak monotonicity asserts that

\[ S(DE) - S(D) + S(EF) - S(F) \geq 0 \]

for any tripartite state \( \rho_{DEF} \).
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Setting \( D = AB, E = C \),

\[ S(CF) - S(F) \geq S(AB) - S(ABC). \]
For any F, $S(CF) - S(F)$ is larger or equal to $S(AB) - S(ABC)$.

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A : C|B)_{\rho} + I(A : C|B)_{\sigma}).$$

$I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC)$
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A : C | B)_{\rho} + I(A : C | B)_{\sigma}).$$

In particular, I can choose F as follows.

$$I(A : C | B) \leq S(CF) - S(F) + S(BC) - S(B)$$
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|^2_1 \leq \frac{1}{2} (I(A : C|B)_{\rho} + I(A : C|B)_{\sigma}).$$

But still, how do you know that $S(CF) - S(F) + S(BC) - S(B)$ is small?

$$I(A : C|B) \leq S(CF) - S(F) + S(BC) - S(B)$$
You don’t. However, given a set of local reduced density matrices, we can easily check this condition. Further, there is a good reason to believe that the upper bound is close to 0 for gapped systems in 1D and 2D.

\[ I(A : C|B) \leq S(CF) - S(F) + S(BC) - S(B) \]
The good reason: strong area law

There is a general belief that if a quantum many-body system has a constant energy gap between its ground state sectors and its first excited state, entanglement entropy satisfies area law:

\[ S(A) = a|\partial A|^{D-1} + b|\partial A|^{D-2} + \ldots . \]

In particular, in 2D,

\[ S(A) = a|\partial A| - \gamma + o(1) \]

(Kitaev and Preskill, Levin and Wen 2006)

* The above assertion is a much stronger statement than this:

\[ S(A) = O(|\partial A|). \]
The upper bound depends on the topology.

Plugging in the entanglement entropy formula,

\[ I(A : C|B) \leq S(CF) - S(F) + S(BC) - S(B) = o(1) \]

\[ I(A : C|B) \leq S(CF) - S(F) + S(BC) - S(B) = 2\gamma + o(1) \]
Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2} (I(A: C|B)_{\rho} + I(A: C|B)_{\sigma}).$$

Plugging in the entanglement entropy formula, we get the desired upper bound.

$$I(A : C|B) \leq S(CF) - S(F) + S(BC) - S(B) = o(1)$$
Suppose two states are consistent over each sites and their neighbours.

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2} (I(A : C | B)_\rho + I(A : C | B)_\sigma).$$

$I(A : C | B) \leq S(CF) - S(F) + S(BC) - S(B) = o(1)$. 

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$$I(A : C | B) \leq S(CF) - S(F) + S(BC) - S(B) = o(1)$$
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$$I(A : C|B) \leq S(CF) - S(F) + S(BC) - S(B) = o(1)$$
Three key ideas

1. If $\rho_{AB} \approx \sigma_{AB}, \rho_{BC} \approx \sigma_{BC}, I(A : C|B)_{\rho} \approx 0$, and $I(A : C|B)_{\sigma} \approx 0$, then $\rho_{ABC} \approx \sigma_{ABC}$.

2. Independent of the size of $A$, there is an upper bound on $I(A : C|B)$ that can be computed from the local reduced density matrices.

3. The upper bound is likely to be small for many interesting systems, e.g., gapped systems in 1D/2D.
Application : Quantum state tomography/verification

1. Quantum state tomography : Estimate the local reduced density matrices, find a state consistent with the local reduced density matrices, and then check the locally computable upper bound. If it is close to 0, we are done!
   
   *Disclaimer:* Finding such a state may take a LONG time.

2. Quantum state verification : Estimate the local reduced density matrices, and check the consistency with the target quantum state. If the locally computable upper bound is close to 0, we are done!
For a large class of interesting multipartite states, there exists a locally checkable condition, under which the expectation values of certain nonlocal observables are completely determined by the expectation values of the local observables.

The condition is likely to be satisfied for generic gapped 1D/2D systems.

For such systems, the number of measurement data that is information-theoretically sufficient to estimate the state grows moderately with the system size.
The technical part of this work is based on the strong subadditivity of entropy and the concavity of von Neumann entropy.

- Better bound using generalized entropies (as opposed to the von Neumann entropy)?

Are there other implications of $I(A : C|B) \approx 0$?

- See 1410.0664 (Fawzi and Renner), 1411.4921 (Brandão et al.), 1412.4067 (Berta et al.), and references therein.

The bound itself is applicable to any quantum states (assuming quantum mechanics is right), and it becomes nontrivial under the strong area law assumption.

- Are there other interesting scenarios under which the bound becomes nontrivial?

For tomographic application, our result does not provide a method to explicitly write down the global state.

- But do we really need to write down the global state when we know that the local data determines the global state?
Globally Computable Upper Bound: Proof Idea

Starting from a useful lemma:
Lemma 1. (Kim 2013)

\[
\frac{1}{8} |\rho - \sigma|_1^2 \leq S\left(\frac{\rho + \sigma}{2}\right) - \frac{S(\rho) + S(\sigma)}{2},
\]

we can show

\[
\frac{1}{8} |\rho_{ABC} - \sigma_{ABC}|_1^2 \leq S\left(\frac{\rho_{ABC} + \sigma_{ABC}}{2}\right) - \frac{S(\rho_{ABC}) + S(\sigma_{ABC})}{2}.
\]

By SSA,

\[
S\left(\frac{\rho_{ABC} + \sigma_{ABC}}{2}\right) \leq S(AB)\frac{\rho + \sigma}{2} + S(BC)\frac{\rho + \sigma}{2} - S(B)\frac{\rho + \sigma}{2}.
\]
Globally Computable Upper Bound: Proof Idea

Starting from a useful lemma:
Lemma 1. (Kim 2013)

$$\frac{1}{8}|\rho - \sigma|^2_1 \leq S\left(\frac{\rho + \sigma}{2}\right) - \frac{S(\rho) + S(\sigma)}{2},$$

we can show

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|^2_1 \leq S\left(\frac{\rho_{ABC} + \sigma_{ABC}}{2}\right) - \frac{S(\rho_{ABC}) + S(\sigma_{ABC})}{2}.$$ 

By SSA,

$$S\left(\frac{\rho_{ABC} + \sigma_{ABC}}{2}\right) \leq S(AB)_{\frac{\rho + \sigma}{2}} + S(BC)_{\frac{\rho + \sigma}{2}} - S(B)_{\frac{\rho + \sigma}{2}}.$$ 

$$S(AB)_{\frac{\rho + \sigma}{2}} = S(AB)_{\rho} = S(AB)_{\sigma},$$ and a similar story for $BC$, $B$. 

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The upper bound depends on the operation.

Plugging in the entanglement entropy formula,

\[ I(A : C|B) \leq S(CF) - S(F) + S(BC) - S(B) = o(1) \]

\[ I(A : C|B) \leq S(CF) - S(F) + S(BC) - S(B) = 2\gamma + o(1) \]

\[ \gamma = \sqrt{\sum_a d_a^2} \quad d_a : \text{quantum dimension of a topological charge } a. \]
Local consistency vs. global consistency depends on $\gamma$ and global topology

- If $\gamma = 0$, an overlapping set of local reduced density matrices completely determine the global state for any compact manifold.
- If $\gamma \neq 0$, an overlapping set of local reduced density matrices completely determine the reduced density matrix over any region that does not contain any logical operator.
  - In particular, an overlapping set of local reduced density matrices completely determine the global state on a sphere.