Einstein-Podolsky-Rosen steering provides the advantage in entanglement-assisted subchannel discrimination with one-way measurements

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“All entangled states are special, but some are more special than others”

George Orwell, *Entanglement farm*
Goals:

- To understand quantum correlations
- To facilitate their exploitation

How:

Operational characterization considering their usefulness in the discrimination of physical processes
initial state  

physical process / transformation

or

final state
We will consider channel with subchannels (a.k.a. instrument) 

\[ \Lambda = \sum_a \Lambda_a \]

\( \hat{\rho} \) : subchannel, i.e. completely positive trace-non-increasing linear map
Includes standard channel discrimination

$$\hat{\Lambda} = \sum_a \Lambda_a \quad \Lambda_a = p_a \hat{\Lambda}_a$$

E.g.: $$\hat{\Lambda} = \frac{1}{2} \hat{\Lambda}_0 + \frac{1}{2} \hat{\Lambda}_1$$

but is more general...
EXAMPLE:

“Branches” of the amplitude damping channel

\[ \hat{\Lambda} = \Lambda_0 + \Lambda_1 \]

\[ \Lambda_i[\hat{\rho}] = K_i \hat{\rho} K_i^\dagger \]

\[ K_0 = |0\rangle\langle 0| + \sqrt{1 - \gamma} |1\rangle\langle 1| \]

\[ K_1 = \sqrt{\gamma} |0\rangle\langle 1| \]
Task:
minimum-error subchannel discrimination

Which $\Lambda_a$?

transformation/evo

$\{\Lambda_a\}_a$

(instrument)
initial state
\( \hat{\rho} \)

transformation/evolution

\( \{ \Lambda_a \}_a \)

(instrument)
\[ p(b, a|\rho) = \text{Tr}(Q_b \Lambda_a [\rho]) \]
Want to optimize the

*probability of guessing correctly*

\[ p_{\text{corr}}(\{\Lambda_a\}_a, \{Q_b\}_b, \hat{\rho}) = \sum_{a,b} p(b, a|\hat{\rho}) \delta_{a,b} \]

\[ = \sum_a \text{Tr}(Q_a \Lambda_a [\hat{\rho}]) \]

same index
Optimal probability of guessing with given input

\[ p_{\text{corr}}(\{\Lambda_a\}_a, \rho) := \max_{\{Q_b\}_b} p_{\text{corr}}(\{\Lambda_a\}_a, \{Q_b\}_b, \rho) \]

Optimal probability of guessing with optimal input

\[ p_{\text{corr}}^{\text{NE}}(\{\Lambda_a\}_a) := \max_{\rho} p_{\text{corr}}(\{\Lambda_a\}_a, \rho) \]
probe
(a.k.a. Bob,
a.k.a. Mario)

ancilla
(a.k.a. Alice,
a.k.a. Luigi)
entangled probe and ancilla

$$\hat{\rho}^{\text{ent}}_{AB} \neq \sum_{\lambda} p(\lambda) \hat{\sigma}_A(\lambda) \otimes \hat{\sigma}_B(\lambda)$$

$$\hat{\sigma}^{\text{sep}}_{AB} \quad \text{separable/unentangled}$$
Optimal probability of guessing with optimal input, including the possibility of using entanglement.

\[ E_{\text{corr}}(\{\Lambda_a\}_a) := \max_{\text{ancilla } A} p^{\text{NE}}_{\text{corr}}(\{\Lambda_a \otimes \text{id}_A\}_a) \]

ancilla does not evolve.
There are evolutions that are better distinguished by the use of entanglement

\[ P^{E}_{\text{corr}}(\{\Lambda_a\}_a) > P^{\text{NE}}_{\text{corr}}(\{\Lambda_a\}_a) \]

[Kitaev, Russ. Math. Surv. ’97; Paulsen, *Completely bounded maps and operator algebras*, ‘02; many others...]
There are evolutions that are **better** distinguished by the use of entanglement

\[ p^E_{\text{corr}}(\{\Lambda_a\}_a) > p^{\text{NE}}_{\text{corr}}(\{\Lambda_a\}_a) \]

[Kitaev, Russ. Math. Surv. ’97; Paulsen, *Completely bounded maps and operator algebras*, ’02; many others...]

**REMARK:**
The classical correlations of *unentangled states are useless!*
There are evolutions that are **better** distinguished by the use of entanglement

**MOREOVER**

For **any** probe-ancilla entangled state, there is a choice of evolutions that are better distinguished using that entangled state

\[
p_{\text{corr}}(\{\Lambda_a(\hat{\rho}_{AB}^{\text{ent}})\}_a, \hat{\rho}_{AB}^{\text{ent}}) > p_{\text{corr}}^{\text{NE}}(\{\Lambda_a(\hat{\rho}_{AB}^{\text{ent}})\}_a)
\]

[P. and Watrous, PRL ‘09]
There are evolutions that are better distinguished by the use of entanglement

MOREOVER

For any probe-ancilla entangled state, there is a choice of evolutions that are better distinguished using that entangled state

Every entangled state is useful for (sub)channel discrimination
resource!
the only resource?
[Matthews, P. and Watrous, PRA ’10]
Does every entangled state stay useful in this scenario?
If measurements are restricted to one-way LOCC, only steerable states can remain useful.

If measurements are restricted to one-way LOCC, all steerable states do remain useful!

The usefulness of a probe-ancilla state in one-way-LOCC subchannel discrimination quantifies its steerability.
Einstein   Podolsky   Rosen
[see above, Phys. Rev. ‘35]

Schroedinger
Alice controls the \textbf{conditional} states of Bob through her choice of measurements.

\[ \rho_{a|x}^B = \text{Tr}_A(M_a^A \rho_{AB}) \]
**EXAMPLE OF STEERING**

\[
\hat{\rho}^{AB} = |\psi^-\rangle\langle\psi^-|^AB
\]

\[
|\psi^-\rangle = \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}
\]

\[
M^A_{a|0} \in \{|0\rangle\langle 0|, |1\rangle\langle 1|\}
\]

\[
\rho^B_{a|0} \in \left\{ \frac{1}{2} |1\rangle\langle 1|, \frac{1}{2} |0\rangle\langle 0| \right\}
\]

\[
M^A_{a|1} \in \{|+\rangle\langle +|, |\rangle\langle -|\}
\]

\[
\rho^B_{a|1} \in \left\{ \frac{1}{2} |\rangle\langle -|, \frac{1}{2} |\rangle\langle +| \right\}
\]

ensembles

assemblage
When is steering really quantum? ("spooky action at a distance")

Can we or can we not imagine that $B$ was in some pre-existing local hidden state?
Local hidden state model

\[
\rho^B_{a|x} = \sum_{\lambda} p(a|x, \lambda)p(\lambda)\hat{\sigma}(\lambda)
\]

\[
\{p(\lambda), \hat{\sigma}^B(\lambda)\}
\]

label of hidden state
probability distribution on hidden states
hidden state
conditional probability

[Wiseman, Jones, Doherty, PRL ‘07]
Local hidden state model

\[
\begin{align*}
\text{label of hidden state} & \quad \{ p(\lambda), \hat{\sigma}^B(\lambda) \} \\
\text{probability distribution on hidden states} & \quad \text{UnSteerable (assemblage)} \\
\rho^{B,\text{US}}_{a|x} & = \sum_{\lambda} p(a|x, \lambda) p(\lambda) \hat{\sigma}(\lambda) \\
& = \sum_{\lambda: \text{det. strat.}} D(a|x, \lambda) p'(\lambda) \hat{\sigma}'(\lambda)
\end{align*}
\]

[Wiseman, Jones, Doherty, PRL ‘07]
Not unsteerable = steerable

A bipartite state is steerable if it can generate steerable assemblages via local measurements; otherwise unsteerable
All unentangled states are unsteerable, and all unsteerable assemblages can be seen as originating from some unentangled state:

steering $\rightarrow$ entanglement

Also some entangled states are unsteerable!!!

[Wiseman, Jones, Doherty, PRL ‘07]
A hierarchy for bipartite correlations
A hierarchy for bipartite correlations
A hierarchy for bipartite correlations
A hierarchy for bipartite correlations
The border we characterize operationally.

all states of Alice and Bob

unsteerable by Alice
\[ \hat{\rho}^{AB} \]  
\[ \rho_{\text{corr}}^{B \rightarrow A}(\{\Lambda_a\}, \rho_{AB}) > \rho_{\text{corr}}^{\text{NE}}(\{\Lambda_a\}) \]  
\[ \{Q_b^{B \rightarrow A}\}_b \]  
\[ \{M^A_{b|x}\}_{b,x} \]
In order to have

\[ p_{\text{corr}}^{B \rightarrow A}(\{\Lambda_a\}, \rho_{AB}) > p_{\text{corr}}^{\text{NE}}(\{\Lambda_a\}) \]

it must be that  \( \{M^A_{b|x}\}_{b,x} \) creates steerable assemblage

(otherwise some separable state would have performed as well, and no better than w/o correlations)
Only steerable states can be useful under the one-way LOCC assumption for measurements
[ also entangled states are useless, if unsteerable!!! ]

We prove that all steerable states do stay useful!!!
If the state is steerable, consider any choice of \( \{ M_{b|x}^A \}_{b,x} \) that generates a steerable assemblage \( \{ \rho_{a|x}^B \}_{a,x} \).

The robustness of steering of such an assemblage is:

\[
R(\{ \rho_{a|x} \}) \\defeq \min \left\{ t \geq 0 \left| \left\{ \frac{\rho_{a|x} + t \tau_{a|x}}{1 + t} \right\}_{a,x} \right. \text{ unsteerable,} \right. \right. \\
\left. \left. \left\{ \tau_{a|x} \right\} \text{ an assemblage} \right\} \right. 
\]
\{\rho_a | x \}

unsteerable assemblages (compatible with unentangled state)
all assemblages

\{ \rho_{a|x} \}\left\{ \frac{\rho_{a|x} + t \tau_{a|x}}{1 + t} \right\}

unsteerable assemblages
(compatible with unentangled state)

\{ \tau_{a|x} \}
We define the **steering robustness of the state** as

\[
R_{\text{steer}}^{A \rightarrow B} (\rho_{AB}) := \sup_{\{M_a^A \}_{a,x}} R(\{\rho^B_{a|x}\}_{a,x})
\]

We prove

\[
\sup_{\{\Lambda_a\}_a} \frac{p_{\text{corr}}^{B \rightarrow A}(\{\Lambda_a\}_a, \rho_{AB})}{p_{\text{corr}}^{\text{NE}}(\{\Lambda_a\}_a)} = R_{\text{steer}}^{A \rightarrow B} (\rho_{AB}) + 1
\]
The direction

\[
\sup_{\{\Lambda_a\}_a} \frac{p^{B\rightarrow A}_{\text{corr}}(\{\Lambda_a\}_a, \rho_{AB})}{p^{\text{NE}}_{\text{corr}}(\{\Lambda_a\}_a)} \leq R^{A\rightarrow B}_{\text{steer}}(\rho_{AB}) + 1
\]

is easily proven just by making use of definitions.

That the upper bound can be achieved is proven by constructing suitable subchannel discrimination problems
Finding $R(\{\rho_{a|x}\})$ corresponds to a **semidefinite programming (SDP)** optimization problem (whose dual is)

$$\text{maximize} \quad \sum_{a,x} \text{Tr}(F_{a|x} \rho_{a|x}) - 1$$

subject to

$$\sum_{a,x} D(a|x, \lambda) F_{a|x} \leq 1 \quad \forall \lambda$$

$$F_{a|x} \geq 0 \quad \forall a, x$$

$D(a|x, \lambda)$ : deterministic response

$\lambda$ : identifier of deterministic response
Entanglement witness

\[ \text{Tr}(W_{AB} \hat{\rho}_{AB}) < 0 \]

\[ \text{Tr}(W_{AB} \hat{\sigma}_{AB}) \geq 0 \]
Steering witness

\[ \{ \rho_{a|x} \} \]

\[ \sum_{a,x} \text{Tr}(F_{a|x} \sigma_{a|x}) \geq 1 \]

all assemblages

unsteerable assemblages
(compatible with unentangled state)

Steering witness

\[ \{ F_{a|x} \} \]
Using the information provided by the SDP optimization problem we construct suitable subchannels \( \{ \Lambda_a \}_a \)

- Choose them to be quantum-to-classical

\[
\Lambda_a [\tau] \propto \sum_x \text{Tr} (F_a | x \tau ) | x \rangle \langle x |
\]

use normalization to make them subchannels of an instrument

from the SDP

orthonormal

- Take care of trace preservation by introducing suitable “dummy” subchannels
Having used the $F_a|x$s that give $R(\{\rho_a|x\})$, with our construction we find

$$\frac{p^{B \rightarrow A}_{corr}(\{\Lambda_a\}_{a}, \rho_{AB})}{p^{NE}_{corr}(\{\Lambda_a\}_{a})} \geq \frac{R(\{\rho_a|x\}) + 1}{1 + \frac{2}{\alpha N}}$$

normalization factor (independent of $N$)

number of dummy subchannels (arbitrary)

Considering $N \rightarrow \infty$ we prove the claim. 🌟
REMARK

Our SDP approach was also inspired by [Skrzypczyk, Navascués, and Cavalcanti, PRL ’14]

In their case they use semidefinite programming to compute the so-called *steering weight*
Steering robustness

\[ \{ \rho_{a|x} \} \rightarrow \left\{ \frac{\rho_{a|x} + t \sigma_{a|x}}{1 + t} \right\} \]

unsteerable assemblages (compatible with unentangled state)

\[ \{ \tau_{a|x} \} \]
Steering weight

\[ \{ \tau_{a|x} \} \]

\[ \{ \rho_{a|x} = (1 - p) \sigma^{US}_{a|x} + p \sigma^{-a|x} \} \]

unsteerable assemblages (compatible with unentangled state)
Conclusions

“All entangled states are special […]”

All entangled states are useful for (sub)channel discrimination

“[…] but some are more special than others”

Only steerable states can be, and are useful for subchannel discrimination under the constraint that the measurements are one-way LOCC
Conclusions

We have introduced the **robustness of steering**:  
- it has at least *two* operational interpretations:  
  - resilience (of steering) to noise  
  - advantage in subchannel discrimination  
- computable via SDP for a given assemblage  
- it provides semi-device-independent bounds to the **robustness of entanglement** [Vidal and Tarrach, PRA ’99]  
- it scales with the amount of entanglement  
- it respects sensible criteria to be considered a resource quantifier [Gallego and Aolita, arXiv:1409.5804]
Some open questions

• Closed formula for the robustness of steering for pure states/maximally entangled states

• Can steering be characterized by considering channel discrimination, rather than subchannel discrimination?

• Are all entangled states useful for (sub)channel discrimination under general LOCC (Vs one-way LOCC)?

• Can we also characterize non-locality --- besides entanglement and steering --- via (sub)channel discrimination tasks?
THANK YOU!!!