Wiring of No-Signaling Boxes Expands the Hypercontractivity Ribbon

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January 12, 2015

Joint work with Amin Gohari
arXiv:1409.3665
ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

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(Received 4 November 1964)
Quantum Nonlocality as an Axiom

Sandu Popescu and Daniel Rohrlich

Received July 2, 1993; revised July 19, 1993

In the conventional approach to quantum mechanics, indeterminism is an axiom and nonlocality is a theorem. We consider inverting the logical order, making nonlocality an axiom and indeterminism a theorem. Nonlocal "superquantum" correlations, preserving relativistic causality, can violate the CHSH inequality more strongly than any quantum correlations.
Closed sets of nonlocal correlations

Limit on Nonlocality in Any World in Which Communication Complexity Is Not Trivial

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(Received 2 March 2006; published 27 June 2006)
Quantum Nonlocality as an Axiom

Sandu Popescu and Daniel Rohrlich

Abstract

Quantum physics has remarkable distinguishing characteristics. For example, it cannot be reduced to any of the classical theories, like classical mechanics, electrodynamic, or quantum mechanics not maximal among those that preserve causality? We give a partial answer to this question by showing that slightly stronger correlations would result in a world in which communication complexity is not trivial.

Limit on Nonlocality in Any World in Which Communication Complexity Is Not Trivial

N. Linden, A. Winter, M. Pawlowski, V. Scarani, and A. Ekert

Abstract

Nonclassical behavior could be produced by quantum-mechanical devices was given by Bell, who proved that no-signalling theories with maximally strong correlations would allow Bob access to all the data in any bit is sent by Alice (to all the data in any other inequality [2], which was easier to translate into a logical structure than entanglement [5].

Here, we call a set of non-local Bell inequalities with Bell coefficients (b, a, c, d) such that b > a and c > d. (Received 2 March 2006; published 25 April 2006)

Quantum Information Causality as a Physical Principle

Marcin Pawłowski, Tomasz Paterek, Dagomir Kaszlikowski, Valerio Scarani, and Andreas Winter

Abstract

The no-signalling (NS) principle is a physical principle: Abstract : N...

Information causality as a physical principle

Marcin Pawłowski, Tomasz Paterek, Dagomir Kaszlikowski, Valerio Scarani, and Andreas Winter

Abstract

We propose the information causality (IC) principle: Abstract : N...

Closed sets of nonlocal correlations

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Closed sets of nonlocal correlations
Local orthogonality as a multipartite principle for quantum correlations

T. Fritz$^{1,2}$, A.B. Sainz$^2$, R. Augusiak$^1$, J. Bohr Brask$^1$, R. Chaves$^{1,3}$, A. Leverrier$^{1,4,5}$ & A. Acín$^{1,6}$

(Received 2 March 2014; published 27 June 2014)
Closed sets of nonlocal correlations

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(Received 11 August 2009; revised manuscript received 27 August 2009; published 11 December 2009)
Outline

- Introduction to non-local boxes and wirings
- Two measures of correlation with the tensorization property
  - Maximal correlation
  - Hypercontractivity ribbon
- Main result: maximal correlation and hypercontractivity ribbon are monotone under wirings
- Example: simulation of isotropic boxes with each other
  - Resolves a conjecture of Lang, Vértesi, Navascués ’14
- Computability of the above invariants
Local measurements on bipartite physical systems

\[ p(a, b | x, y) = \text{the probability of outcomes } a, b \text{ under measurement settings } x, y \]

No-signaling: instantaneous signaling is impossible

\[ p(a | xy) \text{ is independent of } y \]

\[ p(b | xy) \text{ is independent of } x \]
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Local measurements on bipartite physical systems

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Local measurements on bipartite physical systems

- $p(a, b|x, y) =$ the probability of outcomes $a, b$ under measurement settings $x, y$

- **No-signaling**: instantaneous signaling is impossible
  - $p(a|xy)$ is independent of $y$
  - $p(b|xy)$ is independent of $x$
Isotropic boxes

Example: $x, y, a, b \in \{0, 1\}$, and $0 \leq \eta \leq 1$

\[
\text{PR}_\eta(a, b|x, y) := \begin{cases} 
\frac{1 + \eta}{4} & \text{if } a \oplus b = xy, \\
\frac{1 - \eta}{4} & \text{otherwise.}
\end{cases}
\]
Wirings

Wirings are the local operations in the box world. [Allcock et al. '09] The set of physical non-local boxes is closed under wirings. Problem: $1 / 2 \leq \eta' < \eta \leq 1$. Can we generate PR$_{\eta}$ from some copies of PR$_{\eta'}$ under wirings? No if there are two [Short '09] or at most nine [Forster '11] copies of PR$_{\eta'}$ available.
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Wirings are the local operations in the box world

[Allcock et al. ’09] The set of physical non-local boxes is closed under wirings

Problem: $1/2 \leq \eta' < \eta \leq 1$
Can we generate $\text{PR}_\eta$ from some copies of $\text{PR}_{\eta'}$ under wirings?

- No if there are two [Short ’09] or at most nine [Forster ’11] copies of $\text{PR}_{\eta'}$ available
Problem: Given some samples of $p_{AB}$ can we generate one sample from $q_{A'B'}$ under local operations?
Tensorization of measures of correlation

- **Problem:** Given some samples of $p_{AB}$ can we generate one sample from $q_{A'B'}$ under local operations?

- Measures of correlation are monotone under local operations

- $I(A, B)_p < I(A', B')_q \implies \text{No}$
Problem: Given some samples of $p_{AB}$ can we generate one sample from $q_{A'B'}$ under local operations?

Measures of correlation are monotone under local operations

$I(A, B)_p < I(A', B')_q \implies \text{No}$

$\text{NOT quite right!}$

$I(A^n, B^n)_p^n = nI(A, B)_p$.
Tensorization of measures of correlation

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\[ I(A, B)_p < I(A', B')_q \quad \Rightarrow \quad \text{No} \quad \text{NOT quite right!} \]

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- [Tensorization]: Is there a measure of correlation $\rho$ such that

\[ \rho(A^n, B^n)_p^n = \rho(A, B)_p? \]
**Tensorization of measures of correlation**

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  - Maximal correlation
  - Hypercontractivity ribbon
Maximal correlation

Bipartite distribution $p_{AB}$

\[ \rho(A, B) := \max \frac{\text{Cov}(f, g)}{\sqrt{\text{Var}[f_A]\text{Var}[g_B]}} \]

where $f_A : A \rightarrow \mathbb{R}$, $g_B : B \rightarrow \mathbb{R}$.
Maximal correlation

- Bipartite distribution $p_{AB}$

\[
\rho(A, B) := \max \frac{\text{Cov}(f, g)}{\sqrt{\text{Var}[f_A]\text{Var}[g_B]}} \quad f_A : A \to \mathbb{R}, \quad g_B : B \to \mathbb{R}
\]

- $0 \leq \rho(A, B) \leq 1$,  \quad \rho(A, B) = 0 \text{ iff } p_{AB} = p_A \cdot p_B
Maximal correlation

- Bipartite distribution $p_{AB}$

$$\rho(A, B) := \max \frac{\text{Cov}(f, g)}{\sqrt{\text{Var}[f_A]\text{Var}[g_B]}}$$

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- $0 \leq \rho(A, B) \leq 1$, $\rho(A, B) = 0$ iff $p_{AB} = p_A \cdot p_B$

- [Tensorization]: $\rho(A^n, B^n) = \rho(A, B)$

- [Data processing]: $\rho(\cdot, \cdot)$ is monotone under local operations
Maximal correlation

- Bipartite distribution $p_{AB}$
  \[
  \rho(A, B) := \max \frac{\text{Cov}(f, g)}{\sqrt{\text{Var}[f_A] \text{Var}[g_B]}} \quad \text{for } f_A : A \to \mathbb{R}, \ g_B : B \to \mathbb{R}
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- $0 \leq \rho(A, B) \leq 1$, \quad $\rho(A, B) = 0$ iff $p_{AB} = p_A \cdot p_B$

- [Tensorization]: $\rho(A^n, B^n) = \rho(A, B)$

- [Data processing]: $\rho(\cdot, \cdot)$ is monotone under local operations

- Maximal correlation for non-local boxes:
  \[
  \rho(A, B|X, Y) := \max_{x,y} \rho(A, B|X = x, Y = y)
  \]
Lemma: For any no-signaling box $p(ab|xy)$ we have

$$\rho(A, B) \leq \max\{\rho(A, B|X, Y), \rho(X, Y)\}.$$
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$$\rho(A, B) \leq \max\{\rho(A, B|X, Y), \rho(X, Y)\}.$$ 

Proof:

$$\mathbb{E}[fg] = \mathbb{E}_{XY}\mathbb{E}_{AB|XY}[fg]$$

$$\leq \mathbb{E}_{XY}\left[\mathbb{E}_{A|XY}[f] \cdot \mathbb{E}_{B|XY}[g] + \rho \sqrt{\text{Var}_{A|XY}[f] \cdot \text{Var}_{B|XY}[g]}\right]$$

$$= \mathbb{E}_{XY}\left[\mathbb{E}_{A|X}[f] \cdot \mathbb{E}_{B|Y}[g]\right] + \rho \mathbb{E}_{XY}\left[\sqrt{\text{Var}_{A|X}[f] \cdot \text{Var}_{B|Y}[g]}\right]$$

$$\leq \mathbb{E}_{X}\mathbb{E}_{A|X}[f] \cdot \mathbb{E}_{Y}\mathbb{E}_{B|Y}[g] + \rho \sqrt{\text{Var}_{X}\mathbb{E}_{A|X}[f] \cdot \text{Var}_{Y}\mathbb{E}_{B|Y}[g] + \rho \mathbb{E}_{XY}\left[\sqrt{\text{Var}_{A|X}[f] \cdot \text{Var}_{B|Y}[g]}\right]}$$

$$\leq \mathbb{E}_{X}\mathbb{E}_{A|X}[f] \cdot \mathbb{E}_{Y}\mathbb{E}_{B|Y}[g] + \rho \sqrt{\text{Var}_{X}\mathbb{E}_{A|X}[f] \cdot \text{Var}_{Y}\mathbb{E}_{B|Y}[g] + \rho \sqrt{\text{Var}_{X}\mathbb{E}_{A|X}[f] \cdot \text{Var}_{Y}\mathbb{E}_{B|Y}[g]}}$$

$$\leq \mathbb{E}_{AX}[f] \cdot \mathbb{E}_{BY}[g] + \rho \sqrt{\text{Var}_{X}\mathbb{E}_{A|X}[f] + \mathbb{E}_{X}\text{Var}_{A|X}[f]} \cdot \left(\text{Var}_{Y}\mathbb{E}_{B|Y}[g] + \mathbb{E}_{Y}\text{Var}_{B|Y}[g]\right)$$

$$= \mathbb{E}_{AX}[f] \cdot \mathbb{E}_{BY}[g] + \rho \sqrt{\text{Var}_{AX}[f] \text{Var}_{BY}[g]}.$$
Maximal correlation under wirings

Theorem

Maximal correlation of no-signaling boxes does not increase under wirings.
Theorem

Maximal correlation of no-signaling boxes does not increase under wirings.

The proof doesn’t work for these types of wirings! We need new tools.
Hypercontractivity ribbon

- [Ahlswede, Gács ’76] \((\lambda_1, \lambda_2) \in \mathcal{R}(A, B)\) iff

\[
\mathbb{E}[f_A g_B] \leq \|f_A\|_{\frac{1}{\lambda_1}} \|g_B\|_{\frac{1}{\lambda_2}}, \quad \forall f_A, g_B
\]

Schatten norm: \(\|f_A\|_{\frac{1}{\lambda_1}} = \mathbb{E}[|f_A|^{1/\lambda_1}]^{\lambda_1}\)
Hypercontractivity ribbon

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- [Nair ’14] \((\lambda_1, \lambda_2) \in \mathcal{R}(A, B)\) iff:

\[
I(U; AB) \geq \lambda_1 I(U; A) + \lambda_2 I(U; B), \quad \forall p_{U|AB}
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\(\mathcal{R}(A, B) = [0, 1]^2\) iff \(A, B\) are independent
Hypercontractivity ribbon

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- \(\mathcal{R}(A, B) = [0, 1]^2\) iff \(A, B\) are independent

- [Tensorization]: \(\mathcal{R}(A^n, B^n) = \mathcal{R}(A, B)\)

- [Data processing]: \(\mathcal{R}(\cdot, \cdot)\) expands under local operations
Hypercontractivity ribbon for non-local boxes:

\[ \mathcal{R}(A, B|X, Y) := \bigcap_{x, y} \mathcal{R}(A, B|X = x, Y = y). \]
Hypercontractivity ribbon for non-local boxes:

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**Theorem**

Suppose that a no-signaling box \( p(a'b'|x'y') \) can be generated from some copies of a box \( p(ab|xy) \) under wirings. Then

\[ \mathcal{R}(A, B|X, Y) \subseteq \mathcal{R}(A', B'|X', Y'). \]
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Suppose that a no-signaling box \( p(a'b'|x'y') \) can be generated from some copies of a box \( p(ab|xy) \) under wirings. Then

\[ \mathcal{R}(A, B|X, Y) \subseteq \mathcal{R}(A', B'|X', Y'). \]

**Proof:** Chain rule!
Example: Isotropic boxes

\[
\text{PR}_\eta(a, b|x, y) := \begin{cases} 
\frac{1+\eta}{4} & \text{if } a \oplus b = xy, \\
\frac{1-\eta}{4} & \text{otherwise.}
\end{cases}
\]

\[\rho(\text{PR}_\eta) = \eta\]
Example: Isotropic boxes

\[ \text{PR}_\eta(a, b|x, y) := \begin{cases} \frac{1+\eta}{4} & \text{if } a \oplus b = xy, \\ \frac{1-\eta}{4} & \text{otherwise.} \end{cases} \]

\( \rho(\text{PR}_\eta) = \eta \)

Corollary

- For \( 0 \leq \eta' < \eta \leq 1 \), using an arbitrary number of copies of \( \text{PR}_{\eta'} \), a single copy of \( \text{PR}_\eta \) cannot be generated under wirings.
Example: Isotropic boxes

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Corollary

- For \( 0 \leq \eta' < \eta \leq 1 \), using an arbitrary number of copies of \( \text{PR}_{\eta'} \), a single copy of \( \text{PR}_\eta \) cannot be generated under wirings.

- For \( 1/\sqrt{2} \leq \eta' < \eta \leq 1 \), using an arbitrary number of copies of \( \text{PR}_{\eta'} \), a single copy of \( \text{PR}_\eta \) cannot be generated under wirings with shared randomness.
Computation of maximal correlation is easy.
How about computation of the ribbon?

\[ \Upsilon(\cdot) \text{ on the probability simplex by } q_{AB} \mapsto \Upsilon(q_{AB}) = \lambda_1 H(q_A) + \lambda_2 H(q_B) - H(q_{AB}) \]

Let \( \tilde{\Upsilon} \) be the lower convex envelope of distributions.

Lemma
For every distribution \( p_{AB} \), we have \((\lambda_1, \lambda_2) \in \mathbb{R}^{(A, B)}\) if and only if \( \Upsilon(p_{AB}) = \tilde{\Upsilon}(p_{AB}) \).
Computation of maximal correlation is easy. How about computation of the ribbon?

Define $\Upsilon(\cdot)$ on the probability simplex by

$$ q_{AB} \mapsto \Upsilon(q_{AB}) = \lambda_1 H(q_{A}) + \lambda_2 H(q_{B}) - H(q_{AB}) $$

Let $\tilde{\Upsilon}$ be the lower convex envelope of $\Upsilon$ distributions.

Lemma: For every distribution $p_{AB}$, we have $(\lambda_1, \lambda_2) \in \mathbb{R}^{(A, B)}$ if and only if $\Upsilon(p_{AB}) = \tilde{\Upsilon}(p_{AB})$. 
Ribbon in terms of a lower convex envelope

- Computation of maximal correlation is easy. How about computation of the ribbon?
- Define $\Upsilon(\cdot)$ on the probability simplex by
  \[ q_{AB} \mapsto \Upsilon(q_{AB}) = \lambda_1 H(q_A) + \lambda_2 H(q_B) - H(q_{AB}) \]
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Computation of maximal correlation is easy. How about computation of the ribbon?

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$$q_{AB} \mapsto \Upsilon(q_{AB}) = \lambda_1 H(q_A) + \lambda_2 H(q_B) - H(q_{AB})$$

Let $\tilde{\Upsilon}$ be the lower convex envelope of $\Upsilon$

**Lemma**

For every distribution $p_{AB}$, we have $(\lambda_1, \lambda_2) \in \mathcal{R}(A, B)$ if and only if $\Upsilon(p_{AB}) = \tilde{\Upsilon}(p_{AB})$. 

Definition: \((\lambda_1, \lambda_2) \in S(A, B)\) if

\[
\text{Var}[f] \geq \lambda_1 \text{Var}_A \mathbb{E}_{B|A}[f] + \lambda_2 \text{Var}_B \mathbb{E}_{A|B}[f], \quad \forall f_{AB}
\]
**Definition:** \((\lambda_1, \lambda_2) \in \mathcal{G}(A, B)\) if

\[
\text{Var}[f] \geq \lambda_1 \text{Var}_A \mathbb{E}_{B|A}[f] + \lambda_2 \text{Var}_B \mathbb{E}_{A|B}[f], \quad \forall f_{AB}
\]

\[\mathcal{R}(A, B) \subseteq \mathcal{G}(A, B)\]
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\]

- **R(A, B) is a subset of S(A, B)**
- **Tensorization:** \(\mathcal{S}(A^n, B^n) = \mathcal{S}(A, B)\)
- **Data processing:** \(\mathcal{S}(\cdot, \cdot)\) expands under local operations
Maximal correlation ribbon for non-local boxes:

\[
\mathcal{G}(A, B|X, Y) := \bigcap_{x, y} \mathcal{G}(A, B|X = x, Y = y).
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Maximal correlation ribbon for non-local boxes:

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**Theorem**

\[ \rho^2(A, B) = \inf \left\{ \frac{1 - \lambda_1}{\lambda_2} \mid (\lambda_1, \lambda_2) \in \mathcal{S}(A, B) \right\} \]
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Maximal correlation is monotone under wirings.
Summary

- Introduced hypercontractivity ribbon for non-local boxes and showed that it expands under wirings.
- Defined Maximal correlation ribbon.
- Showed that maximal correlation ribbon expands under wirings.
- Characterized maximal correlation in terms of maximal correlation ribbon.
- Maximal correlation is monotone under wirings.
- There is a continuum of closed sets of boxes.
  - Was a conjecture [Lang, Vértesi, Navascués ’14]