On the power of PPT-preserving and non-signalling codes

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A basic problem in quantum information theory is to determine the ability of a noisy channel to convey quantum information at a given standard of fidelity. The *quantum capacity* measures the optimal asymptotic rate of transmission (in qubits per channel use) possible for arbitrarily good fidelities (if not *perfect* fidelity). The LSD (Lloyd [5], Shor [12], Devetak [3]) Theorem shows that the quantum capacity is equal to the regularised coherent information, an optimization that involves unlimited number of copies of the channel. Our understanding of the quantum capacity remains limited – given a simple memoryless channel (such as the qubit depolarizing channel for certain error parameter), determining whether it has a positive quantum capacity is not known to be decidable. To gain insights into the often intractable problem of determining quantum capacities of channels, "assisted capacities" have been studied (see e.g. [1]), where the sender and the receiver are given extra free resources, such as entanglement or classical communication.

In this paper we are interested in the *non-asymptotic* (or finite blocklength) regime focusing on the trade-off between the dimension of the quantum system to be sent, the number of channel uses made, and the fidelity achieved. In the absence of feedback in the coding protocol, this is also called the 'one-shot' regime since we can treat multiple channel uses as a single use of a larger channel. In the one-shot regime, we can remove assumptions such as memoryless channel uses, address questions concerning quantum error correcting codes, and understand how fast the achievable rate converges to the capacity as the number of uses increases. Sometimes, one-shot studies provide results concerning asymptotic capacities. However, the exact trade-off of interest is generally intractable. In the classical case, powerful bounds are known [9]. Parallel to the study of assisted capacities, one can consider assisted codes in the finite blocklength regime.

Mosonyi and Datta [8], Wang and Renner [14] and Renes and Renner [11] have given one-shot converse and achievability bounds for classical data transmission by unassisted codes over classical-quantum channels. In [2] Datta and Hsieh derive converse and achievability results for classical and quantum data transmission by entanglement-assisted codes over general quantum channels in terms of smoothed min- and max-entropies. A drawback of the bounds given in [2] is that no explicit method of computation is given, and it is not clear that an efficient method exists. A one-shot converse bound for entanglement-assisted codes amenable to computation was given in Matthews and Wehner [7] by generalising the hypothesis-testing based 'meta-converse' of [9] to quantum channels. In particular, the bound is a semidefinite program (SDP).

An alternative approach to upper bound one-shot performance is to optimize data transmission over a larger class of coding procedures which is mathematically easier to describe. This approach is applied to entanglement distillation in an early paper by Rains [10], which gives one-shot converse bounds for entanglement distillation by local operations and classical communication in the form of an SDP for the performance of the more powerful class of PPT-preserving operations, along with many other insightful results. This was also the approach used in [6], which derives a linear program for the performance of transmitting classical data via classical channels by codes which are *non-signalling* when the encoder and decoder are considered as a single bipartite operation. The linear program was shown to be equivalent to the meta-converse of [9].

Our work follows this approach. Noting that any *forward-assisted code* corresponds to a bipartite operations which is non-signalling from Bob to Alice, we consider quantum data transmission via quantum channels using codes whose bipartite operation is also non-signalling (from Alice to Bob), PPT-preserving or both. We derive one-shot correspondences that allow our results to be viewed as extensions to results in [7] and [10]. Our main technical contribution is a simple semidefinite programs (SDPs) for the optimal *channel fidelity* of codes which are non-signalling, PPT-preserving, or both. These provide upper bounds on the fidelity of operationally defined classes of codes via the inclusions shown in Figure 1.

Theorem 1. *Let $N_{\mathrm{A'B}}$ be the Choi operator for the channel operation $\mathcal{N}_{\mathrm{B}\leftarrow\mathrm{A'}}$. There is a forward-assisted code of size K, average channel input $\rho_{\mathrm{A'}}$ and channel fidelity f_c for $\mathcal{N}_{\mathrm{B}\leftarrow\mathrm{A'}}$ which is PPT preserving and/or non-signalling from Alice to Bob if and only if there exists an operator $\Lambda_{\mathrm{A'B}}$ such that*

$$f_c = \operatorname{Tr} N_{\mathrm{A'B}}^{\mathrm{T}} \Lambda_{\mathrm{A'B}}, \quad \Lambda_{\mathrm{A'B}} \leq \rho_{\mathrm{A'}} \mathbb{1}_{\mathrm{B}}, \quad \Lambda_{\mathrm{A'B}} \geq 0 \tag{1}$$

$$\mathbf{NS}: \Lambda_{\mathrm{B}} = \mathbb{1}_{\mathrm{B}}/K^2 \tag{2}$$

$$\mathbf{PPTp}: \begin{cases} \mathbf{t}_{\mathrm{B}\leftarrow\mathrm{B}}[\Lambda_{\mathrm{A'B}}] \geq -\rho_{\mathrm{A'}}\mathbb{1}_{\mathrm{B}}/K, \\ \mathbf{t}_{\mathrm{B}\leftarrow\mathrm{B}}[\Lambda_{\mathrm{A'B}}] \leq \rho_{\mathrm{A'}}\mathbb{1}_{\mathrm{B}}/K. \end{cases} \tag{3}$$

Optimising f_c subject to the constraints on line (1) and one or both of constraints (2) and (3), is an SDP. We also give the dual SDPs, for which any feasible point provides an upper bound on the fidelity.

We compare our SDP for the optimal channel fidelity for non-signalling codes with an earlier upper bound for entanglement-assisted codes (derived with different techniques in [7] for the *success probability* of classical data transmission). Surprisingly, our new bound, which applies to a larger class of codes, is at least as tight as the old bound. Furthermore, from the asymptotic analysis of the earlier bound [7], we obtain a new asymptotic result for memoryless noisy channels: that the more powerful class of non-signalling codes yield the same capacity as entanglement-assisted codes.

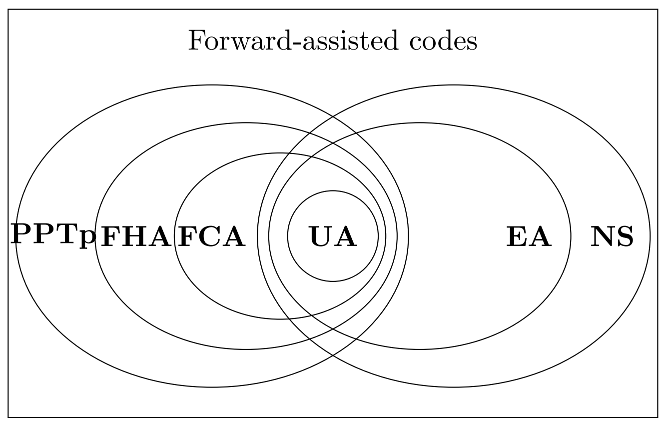

Figure 1: The relationship between various subclasses of forward-assisted codes: PPT-preserving codes **PPTp**; forward-Horodecki-assisted codes **FHA**; forward-classical-assisted codes **FCA**; unassisted codes **UA**; entanglement-assisted codes **EA**; non-signalling codes **NS**;

We also study the optimal channel fidelity for codes which are only PPT-preserving, deriving connections between PPT-preserving codes and PPT-preserving entanglement distillation scheme studied by Rains in [10]. We show that Rains' SDP for the fidelity of PPT-preserving entanglement distillation provides lower bounds on the fidelity of the PPT-preserving codes. We also show that for certain special channels Rains' SDP coincides with our SDP for the fidelity of PPT-preserving codes.

Applying our SDPs to a concrete example, we compute the fidelity for codes that are PPT-preserving, non-signalling or both, over the Werner-Holevo channels for blocklengths up to 120. The results demonstrate that codes which satisfy both constraints can be strictly less powerful than codes that satisfy either one of the constraints. Thus combining the PPT-preserving and

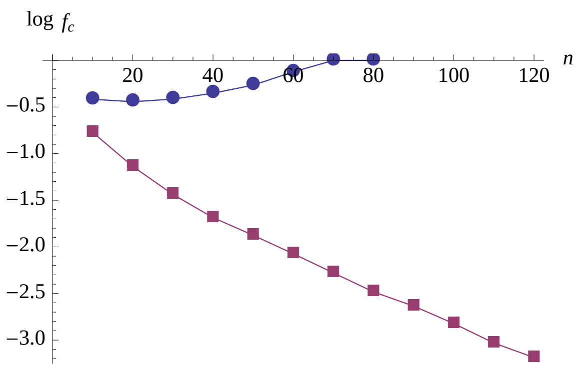

Figure 2: The logarithm (base-two) of the optimal channel fidelity for non-signalling, PPT-preserving codes of size $K_n = \lfloor 2^{rn} \rfloor$ for n uses of the three-dimensional Werner-Holevo channel at rates $r = \log(5/2 - 1/20)$ (circles) and $r = \log(5/2 - 1/40)$ (squares).

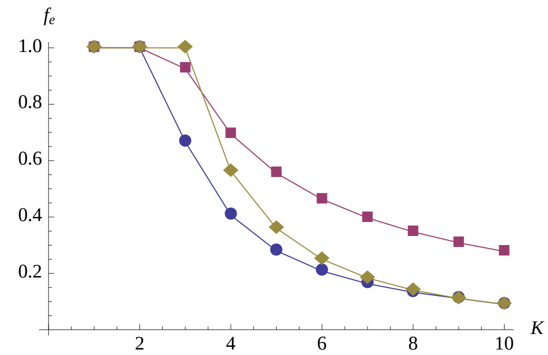

Figure 3: The optimal channel fidelity for sending the state of a K-dimensional system over two-uses of the three-dimensional Werner-Holevo channel using a code which is (i) non-signalling (yellow diamonds), (ii) PPT-preserving (red squares) (iii) both non-signalling and PPT-preserving (blue circles).

non-signalling constraints provides strictly stronger upper bounds for unassisted communication, at least for finite block-lengths. The results suggest that this improvement may even persist in the asymptotic regime.

For $d \geq 3$, the d-dimensional Werner-Holevo channel is anti-degradable and therefore has no quantum capacity. Furthermore, the results of Duan, Severini and Winter [4] imply that it has no entanglement-assisted zero-error classical capacity. Despite this, our results and Rains' [10] imply that PPT-preserving codes enable zero-error quantum communication over the channel at rate $\log(d+2)/d$. Surprisingly, even codes which are both non-signalling and PPT-preserving allow perfect transmission of one qubit over two uses of three-dimensional Werner-Holevo channel (see Figure 3). We discuss the relationship of this phenomenon to the superactivation of quantum capacity [13]. Our result could be considered a form of superactivation, since neither the channel nor the code involved has quantum capacity, yet their combination can communicate quantum data perfectly. However, we do not know whether the code can be implemented by local operations and forward communication over a channel with no quantum capacity. If it could be, then our result would demonstrate a very strong version of superactivation in the sense of [13], where two *channels* with no quantum capacity could be used together to transmit quantum information *perfectly*.

Since any operation which is non-signalling from Bob to Alice can be implemented by forward communication from Alice to Bob, it might be tempting to conjecture that any PPT-preserving operation which is non-signalling from Bob to Alice can be implemented by communication over a PPT-binding channel from Alice to Bob. We show, via an example, this is not the case, but the possibility of the strong form of superactivation remains open.

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