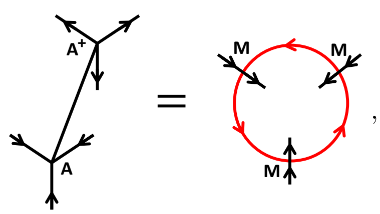
Characterizing Topological Order with Matrix Product Operators - Abstract to [1]

Main Results — In this work, we characterize topological order in projected entangled pair states (PEPS) using matrix product operators (MPO). We propose algebraic rules on MPO and unit PEPS tensor such that PEPS has topological order, i.e., parent Hamiltonian's ground state space depends on the topology of the manifold on which the PEPS is embedded. This set of algebraic rules are satisfied by the previously available G injective and twisted injective PEPS, which were insufficient to describe all string-net models. We furthermore give the explicit MPO and the PEPS for string-net models, and show that they satisfy our algebraic conditions, by giving an intuitive interpretation of pentagon equation in terms of our axioms. This shows that our characterization extends beyond G and twisted injectivity, and contains all string-net models of Levin and Wen. Our approach paves the way to finding novel topological phases beyond string-nets.

Introduction — Topological order at zero temperature is a truly quantum phenomenon which has been discovered in 90's by Wen [2]. The defining features of topological order are the lack of local order parameter [3] and dependence of the ground state degeneracy to the manifold on which the physical system is embedded. In quantum information language, this means that there are pure quantum states having a special type of entanglement pattern such that they cannot be distinguished with local operations. As realized by Kitaev [4], the properties of topological order can be naturally used for fault-tolerant quantum computation.

Characterization of topological order in the ground state is related to identifying entanglement patterns with local indetectability. Some physical restrictions such as local interactions and gap in the Hamiltonian should also be satisfied. In the recent years, a new concept at the interface of quantum information and condensed matter theory, called PEPS (or generally tensor network states) is studied and used [5–7]. PEPS on a lattice can be constructed in the following way: We assign a maximally entangled state to be shared with neighbouring lattice sites, i.e., $\sum_1^D |i\rangle|i\rangle$ where D is called the virtual bond dimension. Then, we apply a linear map A on every site into $\mathbb{C}^d$ where d is called the local physical dimension. As a suitable point of view for topological order, PEPS is a many-body state whose entanglement pattern is determined by the local tensor A. Furthermore, for every PEPS there exists a local, positive-semidefinite, frustration free operator called the parent Hamiltonian whose kernel contains the PEPS. Hence, in the PEPS formalism, characterizing topologically ordered states is equivalent to characterizing the unit PEPS tensor A with suitable axioms so that topological order emerges. For this characterization, we first introduce a new concept called *MPO-injectivity* which generalizes G [8] and twisted injectivitiy [9].

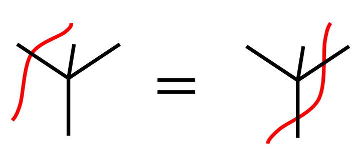
MPO-injectivity — The unit PEPS tensor A can be seen as a map from virtual degrees of freedom to physical ones. To be more specific, let us think about hexagonal lattice, at every vertex the unit PEPS tensor $A : \mathbb{C}^D \otimes \mathbb{C}^D \otimes \mathbb{C}^D \rightarrow \mathbb{C}^d$ sits. We can see this map as a projection onto a subspace $S \in \mathbb{C}^D \otimes \mathbb{C}^D \otimes \mathbb{C}^D$ and then an injective map followed by an isometry to $\mathbb{C}^d$ (generically we need blocking several tensors for above to be completely correct). The idea of MPO-injectivity is that the projector onto the subspace S, i.e., $\mathbb{1}_S$, can be described by MPOs in the following way: By applying the pseudoinverse A^+ to the physical leg of PEPS tensor A we obtain a one-to-one correspondence between virtual and physical degrees of freedom within the subspace S. Pictorially,



(1)

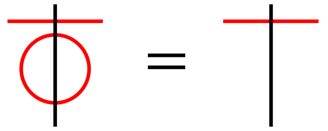
where we make the convention that a PEPS tensor is associated to an intersection of black lines, and a MPO tensor, denoted by M on the right hand side of Eq. (1) where $M : \mathbb{C}^D \otimes \mathbb{C}^m \rightarrow \mathbb{C}^D \otimes \mathbb{C}^m$, to an intersection of a black and a red line (denoting $\mathbb{C}^m$).

Axioms for Topological Order — MPO-injectivity allows us to describe local virtual subspaces in terms of MPOs and it gives a one-to-one correspondence between virtual and physical degrees of freedom. Hence, we can now import the properties of the topologically ordered states to virtual level by stating relevant axioms on MPOs. The first axiom is called *pulling-through*: MPOs are free to pass through the PEPS tensor A on the virtual level



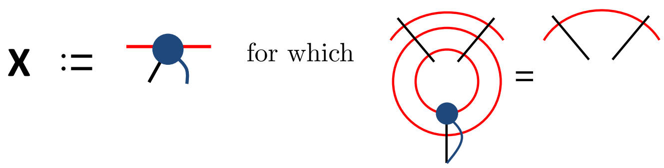
(2)

This condition makes sure that except for at open endpoints, the MPOs are locally unobservable. We also require the following condition for a trivial MPO loop



(3)

which guarantees that any closed MPO is a projector, as required by the definition of MPO injectivity. The final condition required for a consistent definition of the invariant subspace S on arbitrary regions of the lattice is the existence of a tensor X,

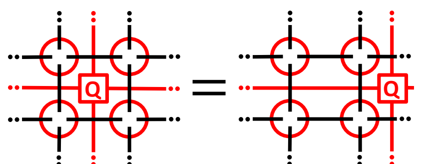


(4)

holds. Together with the pulling through condition [Eq. (2)], Eq. (4) implies that the MPO injectivity is stable when PEPS tensors are concatenated. It further implies that the ground space of the parent Hamiltonian on a contiguous region (which is given by a sum of local terms) is frustration free and is spanned by the concatenated PEPS tensors on the region with arbitrary states on the virtual boundary, which is known as the intersection property.

We conjecture that in two spatial dimensions, all gapped, topologically ordered ground states admitting a PEPS description can be constructed from MPO injective [Eq. (1)] tensors, with the MPO arising as a solution of Eq. (2) and Eq. (3) for which there exists a tensor X satisfying Eq. (4).

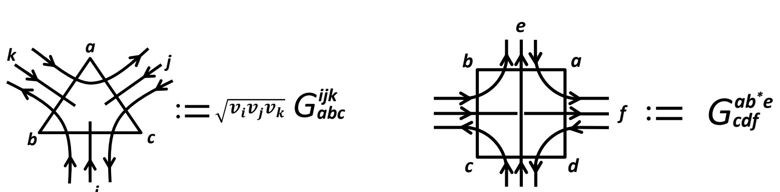
Ground state space — As a result of the algebraic rules, the ground state subspace of a MPO-injective PEPS parent Hamiltonian is spanned by a finite number of states. If the tensor network is closed on a torus, we find that the ground state subspace is spanned by states obtained from tensors Q that satisfy



(5)

where the dots indicate periodic boundary conditions, and the equivalent equation for the vertical direction. The ground state degeneracy on the torus is then given by the number of linearly independent physical states arising from the solution of Eq. (5) that remain distinct and normalizable in the thermodynamic limit. To determine whether any of the resulting degeneracy is truly topological in nature, one must compare the ground state degeneracy of the tensor network on the topological manifold of interest to that arising on a topologically trivial manifold (such as the sphere).

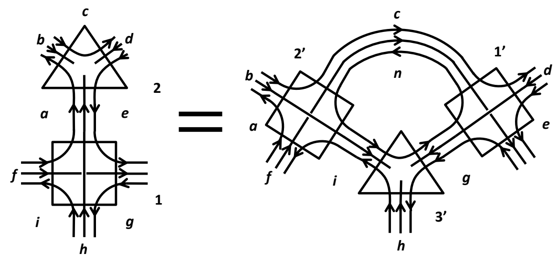
Example: String-nets — We show that the set of models described by MPO-injective PEPS contains all string-net models [10]. In the following, we provide the PEPS tensor [11, 12] on the left and the MPO (that we found) on the right



(6)

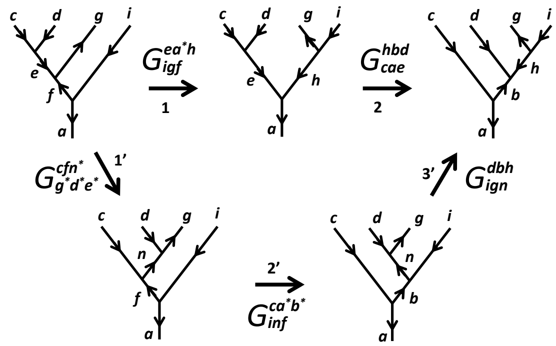
In the above expressions G is a six index tensor, known as the symmetric F-symbol. The MPO for the string-net models turns out to be the G-symbol as well, which is intuitively plausible: The only parameter that defines the string-net condensed state is the G-symbol and the central object in our approach is the MPO. We now show explicitly that for string-nets, our pulling-through axiom (2) is nothing but the pentagon equation, which appears as a consistency condition for the F-symbols [10] and is thus guaranteed to be true for any string-net state. (See original paper [1] for other axioms).

The following is the pulling through condition for the string-net PEPS and MPO tensors



, (7)

which is the pentagon equation pictorially shown by



. (8)

The number labeling each move in the above diagram indicates which of the tensors in Eq. (7) the move corresponds to. The G-symbol next to each move in Eq. (8) is equal to the corresponding tensor in Eq. (7), which can be seen by employing tetrahedral symmetry of the G-symbol.

Closing words — In this work, we propose a set of rules to find topologically ordered PEPS. Compared to few other approaches to topological order, our approach chooses to characterize the overlapping local Hilbert spaces, which is closer to quantum information point of view. We manage this characterization with MPOs and the new concept of MPO-injectivity. This new concept has already given new results for the tensor network community by explaining the string-net models which were not possible before. In our approach, we also cannot see any reason why string-net states are the whole story about bosonic topological order. Indeed, the MPOs give a description of edge physics, because they can be extended to the boundary of the system keeping its one-to-one correspondence to physical edge degrees of freedom. If there are MPOs that lead to protected gapless edge modes, these would describe models beyond string-nets. Furthermore, the framework set forth in this paper can be generalized to fermionic PEPS [13], as well as to higher dimensional systems by replacing MPOs with their higher dimensional generalization, Projected entangled pair operators (PEPOs); it thus provides a systematic way to understand both topological phases of interacting fermions and exotic topological order in three dimensions such as the Haah code [14].

We believe that the classification of intrinsic topological order is also possible with MPOs, similar to results on 2D symmetry-protected topological (SPT) phases [15]. In fact, for string-net models with abelian group elements as local degrees of freedom on edges and with group multiplication as the trivalent vertex constraints, the pentagon equation reduces to a 3-cocycle condition. Examining this connection further may lead to a deep relation between intrinsic and symmetry protected topological phases.

In addition to theoretical contributions, we also foresee contributions to numerics. Our approach is capable of working with models which are not RG-fixed points. Wilson loop operators are in general complicated and spread over the space if the state is not a RG-fixed point. We overcome this problem by carrying everything to virtual level: On the virtual level loop operators (MPOs) are simple and local. The other impact should be in the tensor network contraction algorithms. An equation closely related to the pulling through condition (2) could yield an algorithm to bring 2D PEPS into a normal form that facilitates the calculation of physical observables. Intuitively, this is because once the algorithm has converged we find an MPO that approximates the transfer matrix of the model. Hence, contracting the whole PEPS with a physical observable reduces to contracting the PEPS in a local region around the observable and using a MPO to approximate the boundary.

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