Estimation of group action with energy constraint

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In this talk, we derive the optimal estimation for commutative and non-commutative group with energy constraint. The proposed method can be applied to projective representations of non-compact groups as well as of compact groups. This paper addresses the optimal estimation of $\mathbb{R}$, U(1), SU(2), SO(3), and $\mathbb{R}^2$ with Heisenberg representation under a suitable energy constraint. The technical details are in arXiv:1209.3463 (2012).

Fourier Analytic Approach: In quantum theory, the reversible dynamics of a system is often described by an element in a projective unitary representation of a group. In this case, the unitary acting on the real quantum system reflects important physical parameters. Therefore, we can estimate these physical parameters by estimating the true unitary among a given projective unitary representation of a group. Indeed, it is known that estimation of unitary has a square speed up over the state estimation in quantum case. However, only the limited case of estimation of unitaries has been solved[1–8]. Other case of estimation of unitaries has not been solved while their Fisher information has been calculated[10]. Indeed, several researchers consider that the Fisher information describes the attainable limit of the precision of the estimation of unitary[9–16]. However, as was pointed in [8, 17], it does not give the attainable bound of precision of the estimation of unitary.

The first studies [1, 2] treated the phase estimation, which is essentially the estimation of the representation of U(1). Next, the estimation of SU(2) was studied [3–5]. Chiribella et al [6] established a general theory of estimation of unitary representation of a compact group. Chiribella [18] extended the result to the case of projective representations. Kahn [19] applied this result to the case of SU(d). These studies showed that the estimation error behaves as $\frac{C}{n^2}$ when n is the number of tensor products of the representation. We often call this phenomena the square speed up. For a real implementation, the energy of the input state might be a more important factor than the available number of tensor products. However, many existing studies do not address the optimal estimation with an energy constraint for the input state. This paper deals with this kind of optimization problem.

On the other hand, Imai et al [7] treated phase estimation by using Fourier analysis. In the estimation of action of finite group, the minimum error probability has been shown by [20–22], and that with the projective representation case by [23]. In the case of non-compact groups, the estimation of group action has been formulated by Holevo [24, 25] when the input state is fixed. However, the optimization of input state has been not resolved. That is, there is no general theory of estimation of group action for non-compact groups. In fact, the Fourier transform can be generalized to the case of a non-compact group G, whose generalized version is often called Plancherel transform. In this case, we focus on the set $\hat{G}$ of irreducible representations. Under this method, the input state ϕ can be written as totally square summable (integrable) matrices on irreducible representation spaces. The inverse Fourier transform is given as the unitary operator from the input state ϕ to the square integrable function on G, which can be regarded as an element of $L^2(G)$. Hence, using the inverse Fourier transform, we derive a general optimization result for estimation of a group. In this formula, the minimum error can be written as the minimum of the average error under the distribution given as the square integral of the inverse Fourier transform of the input pure state. Then, we recover existing general results for finite groups and compact groups by [20–23] from our obtained general result.

Further, when the input system is infinite-dimensional, it is natural to restrict the energy of the input state. This constraint is also needed even in the finite-dimensional case. However, the optimal estimation of group action with this type constraint has not been studied sufficiently with a general framework even in the compact case. Using the Fourier transform, this paper gives a general result for this problem. The merit of the obtain general result is to decrease the freedom of optimization. That is, thanks to these results, it is enough to treat the case when the measurement is a specific measurement and the input is pure state. These result reduce our optimization problem to the optimization with respect to input pure states. Further, these results enable us to apply the known result of Fourier analysis because these results clarify the relation with Fourier analysis. The typical obtained results with the energy constraint are summarized as follows although our obtained results cover more general setups.

Estimation of the location sift operation $\mathbb{R}$*:* Firstly, let us consider the estimation of the location sift operation $x \in \mathbb{R}$. In this case, any irreducible representation can be written as $x \mapsto e^{xpi}$ with the momentum $p \in \hat{\mathbb{R}}$ with $\hat{\mathbb{R}} = \mathbb{R}$. Hence, any representation can be written as the unitary $U_x := \int_{-\infty}^{\infty} e^{xpi}|p\rangle\langle p|dp$ on $L^2(\mathbb{R})$. In this case, the input state can be written as a square integrable function ϕ on the momentum space $\hat{\mathbb{R}}$. When we apply the estimator $M(d\hat{x})$, which is a POVM, we obtain the output distribution $\langle\phi|U_x^\dagger M(d)U_x|\phi\rangle$.

Now, we consider the energy constraint on the momentum space $\hat{\mathbb{R}}$ as $\int_{-\infty}^{\infty} p^2|\phi(p)|^2\frac{dp}{\sqrt{2\pi}} \le E$, which can be regarded as a constraint for the kinetic energy. When we adopt the mean square error $\mathcal{D}(M,\phi) := \int_{-\infty}^{\infty}(\hat{x}-x)^2\langle\phi|U_x^\dagger M(d\hat{x})U_x|\phi\rangle$, our problem can be formulated as the minimization problem:

$$\min_{M,\phi}\{\mathcal{D}(M,\phi)|\int_{-\infty}^{\infty} p^2|\phi(p)|^2\frac{dp}{\sqrt{2\pi}} \le E\} = \frac{8}{E}, \tag{1}$$

which can be shown by employing the conventional minimum uncertainty relation. The optima input state is given by a Gaussian wave function. Due to the central limit theorem, the Gaussian wave function can be approximated by the tensor product $\phi^{\otimes n}$ of an arbitrary pure state ϕ. In this case, the optimal coefficient of the first order can be attained by the maximum likelihood estimator with n repeated applications of a proper covariant measurement to the system with the single copy input ϕ.

Estimation of the periodic location sift operation $\mathrm{U}(1)$*:* Next, we consider the estimation of the location sift operation with the periodic condition. In this case, the action can be described as the action $e^{\theta i} \in \mathrm{U}(1)$. Then, any irreducible representation can be written as $\theta \mapsto e^{\theta k i}$ with the momentum $k \in \hat{U(1)}$ with $\hat{U(1)} = \mathbb{Z}$. Hence, any representation can be written as the unitary $U_\theta := \oplus_{k=-\infty}^{\infty} e^{\theta k i}|k\rangle\langle k|$ on $L^2(\mathbb{Z})$. The input state can be written as a square integrable function ϕ on the momentum space $\hat{\mathrm{U}(1)} = \mathbb{Z}$. Now, we consider the energy constraint on the momentum space $\hat{\mathrm{U}(1)}$ as $\sum_{k=-\infty}^{\infty} k^2|\phi(k)|^2 \le E$. Similarly the output distribution is written as $\langle\phi|U_\theta^\dagger M(d\hat{\theta})U_\theta|\phi\rangle$ with the the estimator $M(d\hat{\theta})$. When we adopt the error $\mathcal{D}(M,\phi) := \int_{-\infty}^{\infty}(1-\cos(\hat{\theta}-\theta))\langle\phi|U_\theta^\dagger M(d\hat{\theta})U_\theta|\phi\rangle$, our problem can be formulated as the minimization problem:

$$\min_{M,\phi}\{\mathcal{D}(M,\phi)|\sum_{k=-\infty}^{\infty} k^2|\phi(k)|^2 \le E\} = \max_{s>0}\frac{sa_0(\frac{2}{s})}{4} + 1 - sE \cong \frac{1}{8E} - \frac{1}{128E^2} \text{ as } E \to \infty, \tag{2}$$

where a_0 is a function related to the Mathieu function.

Further, the optimal coefficient of the first order can be attained by the following method. The input state is the tensor product $\phi^{\otimes n}$ of an arbitrary pure state ϕ. We apply a proper covariant measurement to the system with the single copy input ϕ. Finally, we apply the maximum likelihood estimator for n repeated applications of the above measurement.

Estimation of the action $\mathrm{SO}(3)$ *and* $\mathrm{SU}(2)$*:* Next, we consider the estimation of the rotating action $g \in \mathrm{SO}(3)$. In this case, any irreducible representation can be written as $g \mapsto U_{\lambda,g}$ on the irreducible representation space $\mathcal{H}_\lambda$ with the maximum weight $\lambda \in \hat{\mathrm{SO}(3)}$. Hence, any representation can be written as the unitary $U_g := \oplus_{\lambda\in\hat{\mathrm{SO}(3)}} U_{\lambda,g}$ on $\oplus_{\lambda\in\hat{\mathrm{SO}(3)}}\mathcal{U}_\lambda \otimes \mathcal{U}_\lambda^*$, where $\mathcal{U}_\lambda^*$ is the dual space of $\mathcal{U}_\lambda$. In this case, the input state can be written as a square integrable function ϕ on $\oplus_{\lambda\in\hat{\mathrm{SO}(3)}}\mathcal{U}_\lambda \otimes \mathcal{U}_\lambda^*$. When we apply the estimator $M(d\hat{g})$, we obtain the output distribution $\langle\phi|U_g^\dagger M(d\hat{g})U_g|\phi\rangle$.

Now, we consider the energy constraint as $\langle\phi|\oplus_{\lambda\in\hat{\mathrm{SO}(3)}}\lambda(\lambda+1)I_\lambda|\phi\rangle \le E$, where I_λ is the projection to the space $\mathcal{U}_\lambda \otimes \mathcal{U}_\lambda^*$, by using the Casimir operator, which is natural in the relation with the angular momentum. When we adopt the error $\mathcal{D}(M,\phi) := \int_{-\infty}^{\infty}\frac{1}{4}(4-|\mathrm{Tr} g^{-1}\hat{g}|^2)\langle\phi|U_x^\dagger M(d\hat{x})U_x|\phi\rangle$

with use of the gate fidelity $\frac{1}{4}|\mathrm{Tr} g^{-1}\hat{g}|^2$, our problem can be formulated as the minimization problem:

$$\min_{M,\phi}\{\mathcal{D}(M,\phi)|\langle\phi|\oplus_{\lambda\in\hat{\mathrm{SO}(3)}}\lambda(\lambda+1)I_\lambda|\phi\rangle \le E\} = \max_{s>0}\frac{sa_1(\frac{2}{s})}{4}+1-s(E+\frac{1}{4}) \cong \frac{9}{8E}-\frac{81}{128E^2} \quad (3)$$

as $E\to\infty$, where a_1 is a function related to the Mathieu function. Further, the optimal coefficient of the first order can be attained by the method given in the case of U(1). A similar result can be shown when we consider the projective representation of SO(3).

For SU(2), we adopt the error $\mathcal{D}(M,\phi) := \int_{-\infty}^{\infty}(1-\frac{1}{2}\mathrm{Tr} g^{-1}\hat{g})\langle\phi|U_x^\dagger M(d\hat{x})U_x|\phi\rangle$. Then, our problem can be formulated as the minimization problem:

$$\min_{M,\phi}\{\mathcal{D}(M,\phi)|\langle\phi|\oplus_{\lambda\in\hat{\mathrm{SU}(2)}}\lambda(\lambda+1)I_\lambda|\phi\rangle \le E\} = \max_{s>0}\frac{sb_2(\frac{8}{s})}{16}+1-s(E+\frac{1}{4}) \cong \frac{9}{32E}-\frac{7\cdot 3^3}{2^11E^2} \quad (4)$$

a $E\to\infty$, where b_2 is a function related to the Mathieu function.

Estimation of the action of the Heisenberg representation: Finally, we consider the action of the Heisenberg representation $x=(x_1,x_2)\in\mathbb{R}^2$. In this case, the irreducible representation is the equivalent with the Heisenberg representation $x\mapsto U_x$ on $L^2(\mathbb{R})$ when we fix the commutation relation. Then, the input state can be written as a square integrable operator ϕ on $L^2(\mathbb{R})$, which is a pure state on $L^2(\mathbb{R})\otimes L^2(\mathbb{R})$. When we apply the estimator $M(d\hat{x})$, we obtain the output distribution $\langle\phi|U_x^\dagger M(d\hat{x})U_x|\phi\rangle$. Now, we consider the energy constraint as $\langle\phi|(Q^2+P^2)\otimes I|\phi\rangle\le E$. When we adopt the mean square error $\mathcal{D}(M,\phi) := \int_{-\infty}^{\infty}(\hat{x}_1-x_1)^2+(\hat{x}_2-x_2)^2\langle\phi|U_x^\dagger M(d\hat{x})U_x|\phi\rangle$, our problem can be formulated as the minimization problem:

$$\min_{M,\phi}\{\mathcal{D}(M,\phi)|\langle\phi|(Q^2+P^2)\otimes I|\phi\rangle\le E\} = \frac{1}{2E}, \quad (5)$$

which can be shown by reducing the problem to the minimum uncertainty relation on the two-dimensional space.

Uncertainty relations on S^1 and S^3: Using the relation $S^1\cong \mathrm{U}(1)$ and $S^3\cong\mathrm{SU}(2)$, we derive uncertainty relations on S^1 and S^3. Given $\varphi\in L^2(S^1)$, we focus on the relation between $\Delta_\varphi^2(\cos Q,\sin Q) := \Delta_\varphi^2\cos Q+\Delta_\varphi^2\sin Q$ and $\Delta_\varphi^2 P$, where $\Delta_\varphi^2 X := \langle\varphi|X^2|\varphi\rangle-\langle\varphi|X|\varphi\rangle^2$. Then, we obtain

$$\min_{\varphi\in L_n^2(S^1)}\{\Delta_\varphi^2(\cos Q,\sin Q)|\Delta_\varphi^2 P\le E\} = \max_{s>0} 1-(sE-\frac{sa_0(\frac{2}{s})}{4})^2 \cong \frac{1}{4E}-\frac{1}{32E^2} \text{ as } E\to\infty, \quad (6)$$

where $L_n^2(\Omega)$ is the set of normalized functions of $L^2(\Omega)$. Given $\varphi\in L^2(S^3)$, we focus on the relation between $\Delta_\varphi^2\vec{Q} := \sum_{j=0}^3\Delta_\varphi^2 Q_j$ and $\Delta_\varphi^2\vec{P} := \sum_{j=1}^3\Delta_\varphi^2 P_j$, where P_j is the momentum operator for the i-th direction of σ_j via the relation $S^3\cong\mathrm{SU}(2)$. Then, we obtain

$$\min_{\varphi\in L_n^2(S^3)}\{\Delta_\varphi^2\vec{Q}|\Delta_\varphi^2\vec{P}\le E\} = 1-(\min_{s>0} s(E+\frac{1}{4})-\frac{sb_2(\frac{8}{s})}{16})^2 \cong \frac{9}{16E}-\frac{5\cdot 3^3}{2^9E^2} \text{ as } E\to\infty. \quad (7)$$

Conclusion: Our obtained results show that we do not need to use of the entangled input state or the quantum correlation with the quantum measurement to achieve the optimal first order coefficient asymptotically. That is, we only need to use the correlation in the classical data processing because the maximum likelihood estimator employs the correlation in this sense. These results are very contrastive with the square speed-up of estimation of action without energy constraint. Using these result, we have derived uncertainty relations on S^1 and S^3.

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