Local tests of global entanglement and a counterexample to the generalized area law

Dorit Aharonov, Aram W. Harrow, Zeph Landau, Daniel Nagaj, Mario Szegedy, Umesh Vazirani

Abstract

We introduce a technique for applying quantum expanders in a distributed fashion, and use it to solve two basic questions: testing whether a bipartite quantum state shared by two parties is the maximally entangled state and disproving a generalized area law. In the process these two questions which appear completely unrelated turn out to be two sides of the same coin. Strikingly in both cases a constant amount of resources are used to verify a global property.

Introduction. In this paper we address two basic questions:

1. Can Alice and Bob test whether their joint state is maximally entangled while exchanging only a constant number of qubits? More precisely, Alice and Bob hold two halves of a quantum state $|\psi\rangle$ on a D^2-dimensional space for large D, and would like to check whether $|\psi\rangle$ is the maximally entangled state $|\phi_D\rangle = \frac{1}{\sqrt{D}} \sum_x |x\rangle|x\rangle$ or whether it is orthogonal to that state. So far, all known protocols for this task require resources (communication, shared randomness or catalyst) which grow polynomially in $\log(D)$ [3, 2, 8].

2. Is there a counterexample to the generalized area law? A sweeping conjecture in condensed matter physics, and one of the most important open questions in quantum Hamiltonian complexity theory, is the so called Area Law, which asserts that ground states of quantum many body systems on a lattice have limited entanglement. Specifically, assume the system is described by a gapped local Hamiltonian $H = H_1 + \ldots + H_m$, where each H_i describes a local interaction between two neighboring particles of a lattice. The area law conjectures that for every subset S of the particles, the entanglement entropy between S and $\bar{S}$ for the ground state of H is bounded by a constant times the size of the boundary of S. The area law, which has been proven for 1D lattices [9] and is conjectured for higher degree lattices, is of central importance in condensed matter physics as it provides the basic reason to hope that ground states of such systems might have a succinct classical description. The *generalized* area law (a folklore conjecture) transitions from this physically motivated phenomenon to a very clean and general graph theoretic formulation, where in place of edges of the lattice, the terms of the Hamiltonian correspond to edges of an arbitrary graph. It states that for any subset S of vertices (particles), the entanglement entropy between S and $\bar{S}$ for the ground state is bounded by some constant times the cut-set of S (the number of edges leaving S).

We affirmatively answer both questions using a similar technique: applying quantum expanders distributively.

Techniques The main ingredient is the notion of quantum expanders. A quantum expander can be thought of as a collection of d unitaries U_i, (think of d as a constant) each acting on a (possibly large) dimension D Hilbert space. For any X on the D dimensional Hilbert space, $\mathcal{E}(X) = \frac{1}{d}\sum_{i=1}^{d} U_i X U_i^\dagger$ has the unique eigenvalue 1 for the eigenvector $X = \mathbb{I}$ and next highest singular value $\lambda < 1$. It thus shrinks any matrix orthogonal to the identity by a constant factor. The key to the results in the paper is an equivalent way to view quantum expanders, by considering their action on maximally entangled states. It is well known that for any U, $U \otimes U^*$ acting on the maximally entangled state leaves it as is. Of course, this remains true even if U is drawn uniformally at random from the set $U_1, \ldots, U_d$ of the expander. The remarkable fact is that even though quantum expanders use only a constant number d of unitaries, they leave in tact *only* the maximally entangled state, and all other states are shrinked by at least a constant.

For the entanglement testing problem, we use the above intuition to derive a communicating protocol which uses only a constant number of qubits, and detects a maximally entangled state of arbitrary dimension. The idea is that Alice prepares the control state $\sum_{i=1}^{d} |i\rangle|i\rangle$ and sends to Bob one half of this state, which enables them to synchronize which $U_i \otimes U_i^T$ they apply. Bob can then send the register back to Alice, who can then test that the control state remained in tact, implying that the state on which it was applied was not affected, hence, it must have been the maximally entangled state. We derive that for any D, $\epsilon > 0$, there exists a protocol which uses $O(\log 1/\epsilon)$ qubits of communication, after which Bob always accepts if the shared state is $|\phi_D\rangle$. If the shared state is orthogonal to $|\phi_D\rangle$, he accepts with probability at most ϵ. If Alice and Bob do start with the maximally entangled state $|\phi_D\rangle$, the protocol does not damage the state.

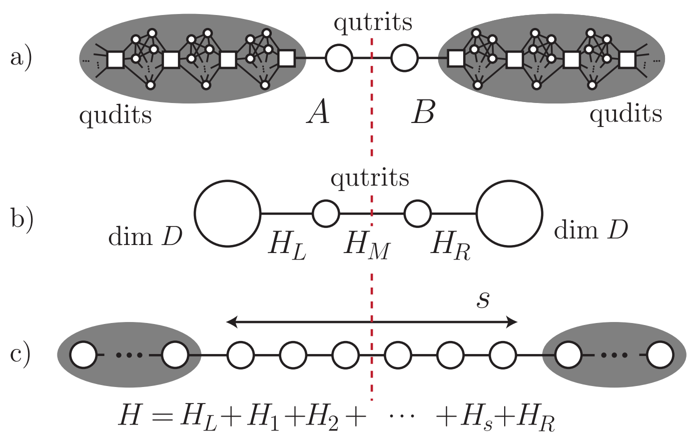

Figure 1: a) A counterexample to the generalized area law, consisting of a chain of complete graphs separated by the middle edge. The entropy across the cut grows as $\Omega(n^c)$, where n is the total number of particles. b) A four-particle construction. c) Short chain framework for proving 1D area law.

For a counterexample to the generalized area law, we use the above intuition to exhibit a gapped local Hamiltonian acting on the graph featured in Figure a, for which the entanglement entropy of the ground state across the middle cut is $\Omega(n^c)$ for some $0 < c < 1$ (rather than $O(1)$ as predicted by the generalized area law). The core step in generating this example is the construction of a simpler system consisting of four particles on a line (see Figure b): two particles of dimension $d = 3$ (qutrits) in the middle, and two particles of dimension D at the two ends, with D is arbitrarily large. The gapped Hamiltonian is of the form $H = H_L + H_M + H_R$, where H_L acts between the left particle and the left qutrit, H_M between the two qutrits, and H_R between the right qutrit and the right particle. Crucially, the entanglement entropy of the ground state across the middle cut is $\Omega(\log D)$.

Like in the communication protocol, we use the middle particles to synchronize the application of the expander on the left and right sides. This requires only a single term of the Hamiltonian, acting on two d-dimensional particles. This four particles example is then converted to a counter example to the generalized area low with bounded dimensional particles (albeit with unbounded degree of interaction) by applying Kitaev's circuit-to-Hamiltonian construction to implement the U_i, followed by an application of the strengthening gadgets of [6].

Discussion and open questions Our two results are reminiscent in spirit to the classical PCP theorem, or more generally to property testing: all those are examples where a small amount of resources (bits checked, Hamiltonian interactions, qubits transmitted, etc.) serve to verify the properties of some large object. However, the fact that such highly non-local properties as global entanglement can be detected using local resources seems rather counter-intuitive. Enforcing of a large amount of entanglement by a single $O(1)$ norm constraint is a surprising quantum phenomenon — we note that in the analogous probabilistic situation, in which we consider the uniform distribution over the set of all possible solutions to constraints set on the graph, the middle constraint can only enforce a convex combination of a *constant* number of product distributions.

What do our results imply regarding the 2D area law, which was indeed the main motivation for this work? The best current $1D$ area law [1] works within a model very similar to our four body Hamiltonian, except the middle link in [1] is extended into a finite chain of $s = \Omega(\log^2 d/\epsilon)$ particles, each of dimension d (see figure c). This yields an area law bound of $S_{1D} = O(\log^3 d/\epsilon)$ across the middle cut. It was observed in [1] that any slight improvement in the exponent of $\log d$ would imply a non-trivial sub-volume law for 2D systems. The crucial parameter in improving the result is the length of the middle chain; Our four body Hamiltonian shows that in the extreme case of a length 1 chain, no area law holds. Understanding the intermediate regime is thus an important open question.

A more modest goal than resolving the $2D$ area law, would be to reduce the degree in our construction to a constant. Such a step already seems to require significant progress in our understanding of related notions, e.g., parallel circuit-to-Hamiltonian constructions (see e.g.,[5]), and quantum expanders which are geometrically constrained.

A possible criticism to our generalized area law counter example is as follows. Should we not expect a generalized area law only in constant degree graphs, where we know that correlations decay exponentially in the groundstate of gapped Hamiltonians? After all, in the $1D$ case the area law follows from exponential decay of correlations [4]. We stress that the connection between exponential decay of corrleations and area laws is in itself merely conjectured in general graphs; it is known to hold only in $1D$ chains. One might indeed view our results as further evidence that such a connection holds also in higher dimensions

Finally, we believe that our results point at a fundamental link between two seemingly unrelated topics. As we show in the paper, it is possible to derive a counterexample to the generalized area law, by starting from an entanglement testing protocol of limited communication, and converting it into a Hamiltonian using Kitaev's circuit-to-Hamiltonian construction. The resulting Hamiltonian can be viewed as a "tester" of its highly entangled groundstate, where the norm of the Hamiltonian terms along the cut corresponds to the communication complexity of the protocol. Whether such a "translation" always exists between entanglement testing protocols of limited communication, and entangled ground states of Hamiltonians with limited interactions between different parts of the system, remains to be explored. Making such an equivalence rigorous might open up a whole new set of tools to studying the area law question, and more generally, help develop better intuition for local Hamiltonians and their groundstates. A related question is whether EPR testing is in fact *equivalent* in some sense to the property of being a quantum expander.

Related work: We remark that our results cannot be derived, to the best of our knowledge, from the results of Gottesman and Hastings [7] and Irani [10], or the improvements of Movassagh and Shor [11]. Those results provide highly entangled states for Hamiltonian whose gaps are inverse polynomial. One might using polynomially stronger interactions in those constructions, and replacing them by weaker interactions using the gadgets of of Nagaj and Cao [6]. This fails since these gadgets, which we would have needed to apply for every edge, introduce a complicated geometry of interactions, and so the size of the cut in the resulting graph would no longer be small.

References

[1] I. Arad, A. Kitaev, Z. Landau, and U. Vazirani. An area law and sub-exponential algorithm for 1D systems. In *Proceedings of the 4th Innovations in Theoretical Computer Science (ITCS)*, 2013, 1301.1162.

[2] H. Barnum, C. Crepeau, D. Gottesman, A. Smith, and A. Tapp. Authentication of quantum messages. In *Foundations of Computer Science, 2002. Proceedings. The 43rd Annual IEEE Symposium on*, pages 449–458, 2002, quant-ph/0205128.

[3] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters. Mixed-state entanglement and quantum error correction. *Phys. Rev. A*, 52:3824–3851, 1996, quant-ph/9604024.

[4] F. G. S. L. Brandão and M. Horodecki. Exponential decay of correlations implies area law. 2012, 1206.2947.

[5] N. P. Breuckmann and B. M. Terhal. Space-time circuit-to-hamiltonian construction and its applications. *Journal of Physics A: Mathematical and Theoretical*, 47(19):195304, 2014.

[6] Y. Cao and D. Nagaj. Perturbative gadgets without strong interactions, 2014. arXiv:1408.5881.

[7] D. Gottesman and M. B. Hastings. Entanglement versus gap for one-dimensional spin systems. *New Journal of Physics*, 12(2):025002, 2010, 0901.1108.

[8] A. W. Harrow and D. W. Leung. A communication-efficient nonlocal measurement with application to communication complexity and bipartite gate capacities. *IEEE Trans. Inf. Theory*, 57(8):5504–5508, 2011, 0803.3066.

[9] M. B. Hastings. An area law for one-dimensional quantum systems. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(08):P08024, 2007, 0705.2024.

[10] S. Irani. Ground state entanglement in one-dimensional translationally invariant quantum systems. *Journal of Mathematical Physics*, 51(2):–, 2010, 0901.1107.

[11] R. Movassagh and P. W. Shor. Power law violation of the area law in critical spin chains, 2014, 1408.1657.