Wiring of No-Signaling Boxes Expands the Hypercontractivity Ribbon

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Abstract: No-signaling boxes are the abstract objects for studying non-locality, and wirings are local operations on the space of no-signaling boxes. This means that, no matter how non-local the nature is, the set of physical non-local correlations must be closed under wirings. Then, one approach to identify the non-locality of nature is to characterize closed sets of non-local correlations. Although non-trivial examples of wirings of no-signaling boxes are known, there is no systematic way to study wirings. In particular, given a set of no-signaling boxes, we do not know a general method to prove that it is closed under wirings. In this paper, we propose the first general method to construct such closed sets of non-local correlations. We show that a well-known measure of correlation, called maximal correlation, when appropriately defined for non-local correlations, is monotonically decreasing under wirings. This establishes a conjecture about the impossibility of simulating isotropic boxes from each other, implying the existence of a continuum of closed sets of non-local boxes under wirings. To prove our main result, we introduce some mathematical tools that may be of independent interest: we define a notion of maximal correlation ribbon as a generalization of maximal correlation, and provide a connection between it and a known object called hypercontractivity ribbon; we show that these two ribbons are monotone under wirings too.

Introduction: Violation of Bell's inequalities is one the most intriguing features of quantum physics. Departure of correlations in the quantum theory from local ones, raises the question of how much non-local the nature is. The Hilbert space formalism of quantum mechanics gives some answers to this question. However, non-locality seems to be a more fundamental feature of nature comparing to postulates of quantum physics. So the question is, can we characterize the limit of non-locality of nature based on more fundamental principles?

This question was first raised by Popescu and Rohrlich [1]. They proposed *no-signaling* as a fundamental physical principle. This is the type of postulate by which one would like to classify the non-locality of nature/quantum mechanics. However, Popescu and Rohrlich showed that no-signaling is not strong enough to characterize non-local correlations of quantum theory. Subsequently, other physical principles were proposed to characterize non-locality [2, 3, 4, 5, 6].

Non-local correlations are generated by locally measuring subsystems of a bipartite system. Imagine that subsystems of a bipartite physical system are held by two parties, say Alice and Bob. Each party can decide to apply a measurement setting; We denote this choice of setting by x for Alice, and y for Bob. Letting the measurement outcomes be a, b, in its full general case, the probability of these outcomes come from some conditional distribution $p(ab|xy)$. We may think of this setting as a *box* with two parts. Each part has an input and an output. Alice who holds the first part can choose its input which we denote by x, and receive its output which we denote by a. Similarly Bob holds the second part, can choose its input y, and receive its output b.

Important examples of no-signaling boxes include *isotropic boxes*. It is a bipartite box with binary inputs and outputs (i.e., $a, b, x, y \in \{0, 1\}$) defined by:

$$\mathrm{PR}_\eta(a, b|x, y) := \begin{cases} \frac{1+\eta}{4} & \text{if } a \oplus b = xy, \\ \frac{1-\eta}{4} & \text{otherwise.} \end{cases} \tag{1}$$

The box PR_η with $0 \le \eta \le 1/2$ is local, and with $0 \le \eta \le 1/\sqrt{2}$ is realizable in quantum mechanics. Nonetheless PR_η for any $0 \le \eta \le 1$ is no-signaling. Thus a natural question is: what is the largest possible η such that PR_η is feasible in nature.

Wirings of no-signaling boxes: Having two boxes, each party can choose the input of the second box as a function of the output of the first box. More precisely, denoting the input and output of the two boxes by subscripts $1, 2$, Alice can first choose x_1 arbitrarily and use the first box to generate an output a_1. Then she may choose x_2 (the input of the second box) as a (possibly random) function of a_1x_1, to obtain the outcome a_2 of the second box. Bob can similarly use the output of the first box to determine the input of the second box. That is, each party can *wire* the output of the first box to the input of the second box. With this *wiring*, the parties generate a new box $p(a_2b_2|x_1y_1)$. That is, combining two boxes with wirings we may generate a new box under local operations.

The wirings of non-local boxes can be more complicated. By the no-signaling condition, the parties can use their available boxes in different orders; Each party can choose an arbitrary ordering of boxes and wire the output of a box to the input of the next box in that order. This point is justified by the no-signaling condition, and can intuitively be verified by viewing local use of boxes as making measurements on subsystems of a bipartite physical system that is available to the parties. Such measurements can be done asynchronously. See Figure 1.

A further degree of freedom in wirings is the very choice of the order of boxes used by a party, as that itself can depend on the outputs of boxes they have already used. For instance, depending on the output of the first box, a party may choose to use the second or third box. Again combining some boxes, the parties can generate a new box under local operations. Wirings indeed are the local operations in the *box world*.

Closed sets of no-signaling boxes: No matter how non-local the nature is, the space of physical boxes must be closed under wirings. This simple observation was first made in [7]. Then to characterize non-locality in nature, we may first look for examples of subsets of no-signaling boxes that are closed under wirings. The set of local (classical) boxes, the set of quantum correlation and the whole set of no-signaling boxes are three examples of such closed sets. As noted in [7] however, finding other examples seems non-trivial. So far, we are only aware of a few *explicit* constructions of sets of no-signaling boxes closed under wirings.

In [8] it is asked whether there exists a continuum of sets of non-local boxes that are closed under wirings. In particular it is conjectured that:

Conjecture 1 ([8]). *For $1/2 < \eta_1 < \eta_2 < 1$, two parties cannot use common randomness and an arbitrary number of copies of PR_{η_1} and with wirings generate a single copy of PR_{η_2}.*

It is shown in [11, 12] that with $n = 9$ copies of PR_{η_1} we cannot generate PR_{η_2} under wirings. Also this problem in the quantum range, i.e., when $\eta_1, \eta_2 \le 1/\sqrt{2}$, is partially answered in [13].

To generalize the above problem, assume that we have two bipartite no-signaling boxes $p(\cdot|\cdot)$ and $q(\cdot|\cdot)$. The question is, can we generate a single copy of $q(\cdot|\cdot)$ using an arbitrary number of copies of $p(\cdot|\cdot)$ with wirings? As far as we know, there is no systematic approach to attack such problems. Such a systematic approach is proposed in our work.

Hypercontractivity ribbon: Let us consider a similar but simpler question of simulating a joint distribution from another. In other words, suppose that we are given two bipartite probability distributions P_{AB} and $Q_{A'B'}$. The question is, given an arbitrary number copies of P_{AB} can we generate a single copy of $Q_{A'B'}$ by only employing local operations? One may ask the same question in the quantum case too. These are hard problems in general, and their difficulty stems from the fact that an arbitrary number of copies of the resource is available.

One may attack this problem by showing that $Q_{A'B'}$ is *more correlated* that P_{AB}, so P_{AB} cannot be transformed to $Q_{A'B'}$ under local operations. This strategy depends on the *measure* of correlation that we use. The point is that we are allowed to use an arbitrary number copies of

P_{AB}. Moreover, for most measures of correlations (including mutual information), if P_{AB} has some positive correlation, the correlation of P_{AB}^n, i.e., n i.i.d. copies of P_{AB}, goes to infinity as n gets larger and larger. This strategy then fails for usual measures of correlation.

Correlation can be measured via hypercontractivity ribbons (HC ribbons) and via maximal correlation. In this abstract, we only define HC ribbon. See [14] for more details. Let A, B be two random variables with joint distribution P_{AB} that take values in finite sets.

Definition 2. *The hypercontractivity ribbon of P_{AB} denoted by $\mathfrak{R}(A, B)$ is the set of pairs of non-negative numbers (λ_1, λ_2) such that for every conditional distribution $P_{U|AB}$ we have*

$$\lambda_1 I(U; A) + \lambda_2 I(U; B) \leq I(U; AB). \tag{2}$$

It can be shown that $\mathfrak{R}(A, B) \subseteq [0, 1]^2$. Moreover, the HC ribbon is equal to $[0, 1]^2$ if and only if A, B are independent.

Theorem 3. *The HC ribbon has the following properties:*

(i) [Tensorization] *If $P_{A_1A_2B_1B_2} = P_{A_1B_1}P_{A_2B_2}$, then $\mathfrak{R}(A_1A_2, B_1B_2) = \mathfrak{R}(A_1, B_1) \cap \mathfrak{R}(A_2, B_2)$.*

(ii) [Data processing] *If $P_{A_1A_2B_1B_2} = P_{A_1B_1}P_{A_2|A_1}P_{B_2|B_1}$, then $\mathfrak{R}(A_1, B_1) \subseteq \mathfrak{R}(A_2, B_2)$.*

Part (i) in particular implies that letting A_iB_i, $i = 1, \ldots, n$, be n i.i.d. copies of AB then

$$\mathfrak{R}(A_{[n]}, B_{[n]}) = \mathfrak{R}(A, B).$$

Part (ii) means local transformations on individual random variables can only *expand* the HC ribbon. This is in line with the fact that HC ribbon is the whole $[0, 1]^2$ for independent random variables.

The standard definition of HC ribbon [9] is in terms of Schatten norms of functions of random variables, rather than mutual information. A remarkable recent work by Nair [10] finds a representation of the HC ribbon for two random variables in terms of mutual information.

HC ribbon for no-signaling boxes: We define the hypercontractivity ribbon for no-signaling boxes, and show that it is well-behaved under wirings.

Definition 4. *Given a no-signaling box $p(ab|xy)$, we define its HC ribbon to be the intersection of the HC ribbons of its outputs conditioned on all possible inputs, i.e.,*

$$\mathfrak{R}(A, B|X, Y) := \bigcap_{x,y} \mathfrak{R}(A, B|X = x, Y = y).$$

The main technical result of our work is the following theorem.

Theorem 5. *Suppose that a no-signaling box $p(a'b'|x'y')$ can be generated from n no-signaling boxes $p_i(a_ib_i|x_iy_i)$ where $i \in [n]$, under wirings. Then we have*

$$\bigcap_{i=1}^{n} \mathfrak{R}(A_i, B_i|X_i, Y_i) \subseteq \mathfrak{R}(A', B'|X', Y'). \tag{3}$$

A similar theorem for maximal correlation is also valid (see [14]). Using this theorem we can systematically construct sets of no-signaling boxes closed under wirings.

Corollary 6. *Let $\Lambda \subseteq [0, 1]^2$ be an arbitrary subset. Then the set of no-signaling boxes whose HC ribbon contains Λ is closed under wirings.*

Example: isotropic boxes: As an application of Theorem 5 we prove Conjecture 1 in a certain range of parameters.

Theorem 7. *The followings hold.*

- *For $0 \leq \eta_1 < \eta_2 \leq 1$, using an arbitrary number of copies of PR_{η_1}, a single copy of PR_{η_2} cannot be generated under wirings.*
- *For $1/\sqrt{2} \leq \eta_1 < \eta_2 \leq 1$, using common randomness and an arbitrary number of copies of PR_{η_1}, a single copy of PR_{η_2} cannot be generated under wirings.*

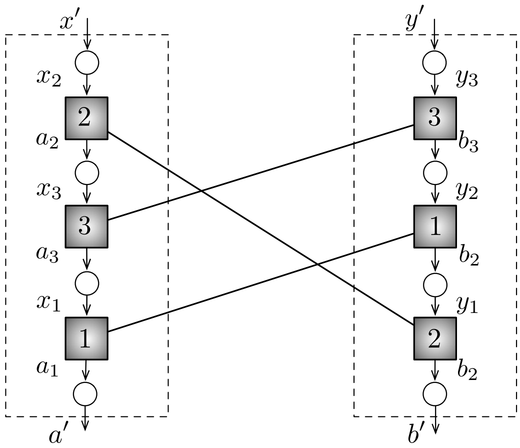

Figure 1: Wiring of three no-signaling boxes

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