# A non-commuting Stabilizer Formalism

Xiaotong Ni*, Oliver Buerschaper†, and Maarten Van den Nest*

*Max-Planck-Institut für Quantenoptik, Garching, Germany
†Perimeter Institute for Theoretical Physics, Waterloo, Canada

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**Abstract**

We propose a non-commutative extension of the Pauli stabilizer formalism (see [1] for the full article). The aim is to describe a class of many-body quantum states which is richer than the standard Pauli stabilizer states. In our framework, stabilizer operators are tensor products of single-qubit operators drawn from the group $\langle \alpha I, X, S\rangle$, where $\alpha = e^{i\pi/4}$ and $S = \mathrm{diag}(1, i)$. We provide techniques to efficiently compute various properties, related to e.g. bipartite entanglement, expectation values of local observables, preparation by means of quantum circuits, parent Hamiltonians etc. We also highlight significant differences compared to the Pauli stabilizer formalism. In particular we give examples of states in our formalism which cannot arise in the Pauli stabilizer formalism, such as topological models that supports non-abelian anyons.

**1. Motivation.** The Pauli stabilizer formalism (PSF) is a widely used tool throughout the development of quantum information (e.g. quantum error correcting codes, measurement based quantum computation, toric code). At the same time it has a simple underlying group picture, which makes it a powerful tool to construct new codes and physical models [2, 3].

A more general way to describe quantum states and subspaces is by using commuting projectors. For example, most physical models with topological order can be described this way. Many properties of commuting projector codes are known [4–6]. However, it is not an easy task to construct interesting examples with commuting projectors. This difficulty partially stems from the computational complexity: while for Pauli stabilizer groups it is always efficient to find stabilized states or subspaces, in general the commuting projector problems are NP-hard [7].

Based on the above consideration, it is very desirable to have an intermediate class between the PSF and commuting projector codes. In this work we propose one such generalization of the PSF, which is able to describe non-additive codes and more physical models, while at the same time has a relatively simple structure.

**2. Definition** In the Pauli stabilizer formalism, we consider stabilizer operators that are tensor products of the single-qubit Pauli group $\langle iI, X, Z\rangle$. In this paper, we generalize it to tensor products of $\langle \alpha I, X, S\rangle$ where $\alpha = e^{i\pi/4}$ and $S = \mathrm{diag}(1, i)$. Thus each stabilizer operator is a tensor product $g = g^{(1)} \otimes \cdots \otimes g^{(n)}$ where each $g^{(i)}$ is an element of $\langle \alpha I, X, S\rangle$. It is easy to to show that every such stabilizer operator can be written as

$$g = \alpha^s X^{a_1} S^{b_1} \otimes \cdots \otimes X^{a_n} S^{b_n} \equiv \alpha^s X(\vec{a}) S(\vec{b}), \tag{1}$$

where $s \in \{0, \dots, 7\}$, $a_j \in \{0, 1\}$, $b_j \in \{0, 1, 2, 3\}$, and $\vec{a} = (a_1, \dots, a_n)$, $X(\vec{a}) = X^{a_1} \otimes \cdots \otimes X^{a_n}$. $\vec{b}$ and $S(\vec{b})$ are defined similarly. For a set of such operators $g_1, \dots, g_m$, we consider the group

$G = \langle g_1, \ldots, g_m \rangle$, and we say a state $|\psi\rangle$ is stabilized by $G$ if for every $g \in G$, we have $g|\psi\rangle = |\psi\rangle$. The space of all $|\psi\rangle$ stabilized by $G$ will be called the XS-stabilizer code associated with $G$. If $|\psi\rangle$ is uniquely stabilized by $G$, we call this state an XS-stabilizer state. Since $S^2 = Z$, it is clear the $XS$-stabilizer formalism is a generalization of the Pauli stabilizer formalism. One major difference is that in the $XS$-stabilizer formalism, it may happen that XS-stabilizer states/codes have a non-abelian XS-stabilizer group $G$ – while Pauli stabilizer groups must always be abelian.

## 3. Examples.

A drastic difference between Pauli and $XS$-stabilizer codes is the power of translational invariant stabilizers in 2D. We show that all twisted quantum double model $D^\omega(\mathbb{Z}_2^n)$ have an $XS$-stabilizer description (up to a local unitary transformation). One example of that family is the doubled semion model, which is an instance of the string-net model family [8]. Many of its properties are very close to the toric code, while they support different abelian anyons. More Interestingly, it is known that some of the models in $D^\omega(\mathbb{Z}_2^n)$ support non-abelian anyons. Based on the results in [9, 10], we know that these models cannot be described by 2D (translational invariant) Pauli stabilizer. This illustrates that the XS-stabilizer formalism can be used to describe significantly different states compared to the Pauli stabilizer formalism.

More examples can be found in the full article [1]. We also characterize all possible XS-stabilizer states.

## 4. Properties of $XS$-stabilizer states

- *Computational complexity of finding stabilized state.* In the Pauli stabilizer formalism, it is always computationally easy to determine whether, for a given set of stabilizer operators, there exists a common stabilized state. However, we will prove that the same question is **NP**-complete for $XS$-stabilizers. More precisely, given a set of XS-stabilizer operators $\{g_1, \ldots, g_m\}$, deciding whether there exists a state $|\psi\rangle$ stabilized by each of the $g_j$ is proved to be **NP**-complete. The NP-hardness part is proved by reducing the 1-IN-3 SAT problem to our problem. To show that the problem is inside **NP**, we use tools developed to analyze monomial stabilizers, as developed in [11].

  The NP-hardness partially stems from the fact that the group $G = \langle g_1, \ldots, g_m \rangle$ contains diagonal operators which have $S$ within the tensor product. To deal with this, we impose a constraint on the group $G$, namely every diagonal operator in $G$ can be written as a tensor product of $I$ and $Z$ (in other words $\pm Z(\vec{b})$). Given this constraint, the existence of states stabilized by $G$ is then equivalent to the condition $-I \notin G$, which can be checked efficiently (and is in fact the same as for Pauli stabilizers). We call such group $G$ a *regular* $XS$-stabilizer group.

  We will show that in fact all XS-stabilizer states (i.e. those uniquely stabilized by a group $G$) afford regular stabilizer groups. This is also the case for many XS-stabilizer codes, including the doubled semion model.

- *Entanglement.* Given an $XS$-stabilizer state $|\psi\rangle$ with associated XS-stabilizer group $G$, we show how to compute the von Neumann entropy for any bipartition $(A, B)$. This is achieved by showing that $|\psi\rangle$ can always be transformed into a Pauli stabilizer state by applying a unitary $U_A \otimes U_B$, where $U_A$ and $U_B$ each only act on the qubits in one party. This implies in particular that, for any subset of qubits, the reduced density matrix $\rho$ is always be a projector, since this is the case for Pauli stabilizer states. Thus for XS-stabilizer states, all $\alpha$-Renyi entropies coincide. It is worth noting that our proof uses a very different technique compared to the one typically used to study the entanglement of Pauli stabilizer states (for example, the methods in [12]).

We also formulate the following open problem: for every XS-stabilizer state, does there always exist a Pauli stabilizer state $|\phi\rangle$ having the same Schmidt rank as $|\psi\rangle$ for every bipartition? For example, it would be interesting to know whether the inequalities in [12] holds for $XS$-stabilizer states.

- *Commuting parent Hamiltonian.* Even though an XS-stabilizer group $G$ is in general not abelian, we show that we can always find a commuting Hamiltonian $H = \sum_j H_j$ which has a ground space identical to the space stabilized by $G$. If the generators $\{g_j\}$ of $G$ satisfy some locality condition (e.g. local on a 2D lattice), then $H_j$ will satisfy the same locality condition (up to a constant factor). This means that general properties of states described by commuting projectors also apply to XS-stabilizer states. For example, for a state $|\psi\rangle$ uniquely stabilized by $XS$-stabilizer on $\mathcal{D}$ dimensional lattice, it would satisfy the area law [13], and for $XS$-stabilizer on 2D lattice, we can find string like logical operators [4]. However, note that in the above discussion we assume we are only interested in the ground space of the XS-stabilizer codes and commuting projector codes. It is possible that the non-commuting parent Hamiltonian of a XS-stabilizer code has a very different behaviour compared to the commuting one.

- *Efficient algorithms.* We show that for any XS-stabilizer state $|\psi\rangle$ with regular $XS$-stabilizer group $G$, the following tasks can be done efficiently (in polynomial time in the number of qubits):

  1. Compute von Neumann entropy for any bipartition.
  2. Compute expectation values of local observables.
  3. Prepare $|\psi\rangle$ on a quantum computer by means of a poly-size quantum circuit.
  4. Compute the function $f(x)$ in the standard basis expansion

$$|\psi\rangle = \sum_x f(x)|x\rangle. \tag{2}$$

Moreover, for a general XS-stabilizer code (i.e. degenerate subspace) with regular $XS$-stabilizer group $G$, we can efficiently construct a set of basis $|\psi_1\rangle,\dots,|\psi_k\rangle$. For each $|\psi_j\rangle$ we can again perform all the above tasks efficiently.

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