Gauge Color Codes

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Transversal gates and spatial dimension

In fault-tolerant quantum computation [1], quantum information is protected from noise by encoding it in somewhat non-local degrees of freedom, thus distributing it among many smaller subsystems, typically qubits. This makes sense under the physically relevant assumption that interactions with the environment have a local nature. The implementation of gates, consequently, must be also as local as possible to preserve the structure of noise. This is naturally achieved with transversal gates, *i.e.* unitary operators that transform encoded states by acting separately on suitable subsystems, in the simplest case independently on each qubit. Unfortunately, no code admits a universal transversal set of gates [2].

Topological quantum error correcting codes [3] introduce a richer notion of locality by considering the spatial location of the physical qubits, which are assumed to be arranged on a lattice. They come in families parametrized with a lattice size, for a fixed spatial dimension. Their defining features are (i) that the measurements needed to recover information about errors only involve a few neighbouring qubits and (ii) that no encoded information can be recovered without access to a number of physical qubits comparable to the system size.

Rather than sticking to the above definition of transversality, for topological codes it is natural to consider instead quantum circuits of fixed depth with geometrically local gates [4, 5]. Remarkably, it has been recently shown that spatial dimension imposes constraints on the transversal gates that are allowed [5]. In particular, this is true in the case of topological stabilizer codes, where it has been found that transversal gates on D-dimensional codes have to belong to certain sets $\mathcal{P}_D$.

A first main result of this work [6] is that codes that saturate these bounds can be constructed for every D, in the sense that they admit at least a transversal gate in $\mathcal{P}_D$ (not belonging to $\mathcal{P}_{D-1}$). In particular, the result applies to color codes [7, 8, 9, 10], where the single-qubit gate $R_D := \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^D} \end{smallmatrix}\right)$ is transversal (in the more strict non-topological sense).

It was already known that color codes could have this property as long as lattices with certain local properties existed [10], but no recipe was known to construct suitable lattices. The new result is that in fact such local constraints are not relevant, suggesting an interesting interpretation. Topological stabilizer codes can be regarded as ground states of gapped local Hamiltonians: they give examples of a form of quantum order in condensed matter, namely topological order [11]. Different topological orders are characterized by the topological interactions of their excitations. Given the (topological) notion of transversal gate, it is natural to expect that each kind of topological order should allow a different set of transversal gates, irrespective of other *local* specific details of the system. Then it is not surprising that all color codes of a given dimensionality (for certain boundary conditions!) have the same transversal gates, as they all have the same topological order.

On the practical side, less constraints on the choice of the lattice of physical qubits implies that more efficient color codes are possible. Most important are lower dimensions, and in particular 2D color codes, for which the whole Clifford group is transversal. Without the constraints, it becomes possible to put the qubits in a honeycomb lattice: error syndromes are recovered by measuring 6 qubit operators with the geometry of the hexagons. In contrast, with the constraints lattices involving octagons where necessary [7]. This simplification should result in a very significant improvement of the error threshold and the qubit overhead, making color codes into an even stronger alternative to toric codes for implementation.

Gauge fixing and universal transversal gate sets

3D color codes are also very interesting: CNot and R_3 gates are transversal, so that the Hadamard gate suffices to complete a universal gate set. The latter in turn can be recovered if $|+\rangle$ encoded states are available [12], but these can be easily created since color codes are CSS codes. From a practical perspective, however, 3D color codes pose two difficulties. The main one is that the measurements for error-correction can involve each dozens of qubits; this is a problem because, generally speaking, operations involving more qubits tend to be more unreliable and lengthy. In addition, each Hadamard gate requires an extra encoded qubit, thus increasing the resources needed for computation.

Both problems disappear in this work [6] by introducing a subsystem form of color codes, *gauge color codes* [13]. Recall that in subsystem codes the code space contains both logical and gauge qubits. The latter are just qubits that we do not care about, and in a topological code they might include local degrees of freedom. In the case of 3D gauge color codes, an important aspect is that the code space of the corresponding conventional 3D color code is contained in the code space of the gauge code. Moreover, the logical operators coincide, so that one can jump back and forth between the two codes by either fixing the gauge degrees of freedom (gauge $\rightarrow$ conventional) or doing nothing (conventional $\rightarrow$ gauge). Thus a *universal* set of transversal gates becomes available! All thanks to the *gauge fixing* technique [14], which elegantly dodges the no-go theorem on transversal gates.

As anticipated above, the problem of measuring operators involving dozens of qubits also disappears. The reason is that instead of measuring directly the relevant operators, alternatively one can measure gauge operators that only involve 4 or 6 qubits each. As a byproduct, such measurements give redundant information that can be put to use in the presence of measurement errors [15].

Summary

Local codes receive increasing attention because of their potential practical applications. Among them 3D gauge color code are quite singular: they allow a universal transversal set of gates via gauge fixing, all the needed measurements involve up to 6 qubits and, surprisingly [15], the redundancy involved in recovering the error syndrome allows to perform all fault-tolerant operations in constant time (disregarding classical computations). For these reasons, 3D gauge color codes should have a significant impact in future developments and could even be of direct practical interest.

References

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