Fault-tolerant logical gates in quantum error-correcting codes*

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Abstract

Recently, Bravyi and König have shown that there is a trade-off between fault-tolerantly implementable logical gates and geometric locality of stabilizer codes. They consider locality-preserving operations which are implemented by a constant-depth geometrically-local circuit and are thus fault-tolerant by construction. In particular, they shown that, for local stabilizer codes in D spatial dimensions, locality preserving gates are restricted to a set of unitary gates known as the D-th level of the Clifford hierarchy. In this paper, we elaborate this idea and provide several extensions and applications of their characterization in various directions.

First, we present a new no-go theorem for self-correcting quantum memory [17]. Namely, we prove that a three-dimensional stabilizer Hamiltonian with a locality-preserving implementation of a non-Clifford gate cannot have a macroscopic energy barrier. This result implies that in Haah's Cubic code [18] and Michnicki's [19] welded code non-Clifford gates do not admit such an implementation.

Second, we prove that the code distance of a D-dimensional local stabilizer code with non-trivial locality-preserving m-th level Clifford logical gate is upper bounded by $O(L^{D+1-m})$. For codes with non-Clifford gates ($m > 2$), this improves the previous best bound by Bravyi and Terhal. Bombin and Martin-Delgado's topological color codes saturate the bound for $m = D$.

Third we prove that a qubit loss threshold of codes with non-trivial transversal m-th level Clifford logical gate is upper bounded by $1/m$. As such, no family of fault-tolerant codes with transversal gates in increasing level of the Clifford hierarchy may exist. This result applies to arbitrary stabilizer and subsystem codes, and is not restricted to geometrically-local codes.

Finally, we extend the result of Bravyi and König to subsystem codes. A technical difficulty is that, unlike stabilizer codes, the so-called union lemma does not apply to subsystem codes. This problem is avoided by assuming the presence of error threshold in a subsystem code, and the same conclusion as Bravyi-König is recovered.

Quantum error-correcting codes constitute an indispensable ingredient in the roadmap to fault-tolerant quantum computation as they offer the framework of enabling imperfect quantum gates and resources to implement arbitrarily reliable quantum computation [2, 3]. An essential feature for such codes is to admit a fault-tolerant implementation of a universal gate-set where physical errors should propagate in a benign and controlled manner. A paragon for fault-tolerant implementation of logical gates is provided by transversal unitary operations, *i.e.* single qubit rotations acting independently on each physical qubit.

However, Eastin and Knill have proved that the set of transversal gates constitutes a finite group, and hence is not universal for quantum computation [4], suggesting a tension between computational power and fault-tolerance. Recently, Bravyi and König have further sharpened this tension for topological stabilizer codes supported on a lattice with geometrically local generators [5]. By extending their consideration to logical gates implemented by constant depth local quantum circuits as feasible proxy, they have shown that, in D spatial dimensions, fault-tolerantly implementable logical gates

*This extended abstract summarizes [1].

are restricted to a set of unitary gates, known as the D-th level of the Clifford hierarchy [6]. This result establishes a connection between two seemingly unrelated notions; fault-tolerance and geometric locality.

The result by Bravyi and König (BK) is motivated by considerations of topological stabilizer codes, which are also likely to suggest a host of future generalizations. In this paper, we begin to address open questions posed by the work of Bravyi and König.

Clifford hierarchy.- As in BK [5], the tensor product Pauli operators on n qubits (denoted by $\mathsf{Pauli} = \langle X_j, Y_j, Z_j \rangle_{j\in[1,n]}$) and the corresponding Clifford hierarchy [6] will play a central role. We provide a formal definition for the m-th level of the Clifford hierarchy $\mathcal{P}_m$.

Definition 1. *We define the **Clifford hierarchy** as $\mathcal{P}_0 \equiv \mathbb{C}$ (i.e. global complex phases), and then recursively as*

$$\mathcal{P}_{m+1} = \{U : \forall P \in \mathsf{Pauli},\ UPU^\dagger P^\dagger \in \mathcal{P}_m\}. \tag{1}$$

Note that despite using a commutator in place of conjugation, the above definition coincides with the usual one for $m \geq 2$ [6, 5]. $\mathcal{P}_1$ is a group of Pauli operators with global complex phases. $\mathcal{P}_2$ coincides with the *Clifford group* and includes the Hadamard gate H, $\pi/2$ phase shift and the CNOT gate. $\mathcal{P}_3$ includes some non-Clifford gates such as $\pi/4$ phase shift and the Toffoli gate. $\pi/2^{m-1}$ phase shift belongs to $\mathcal{P}_m$. Note that $\mathcal{P}_m$ is a set and is not a group for $m \geq 3$.

The Gottesman-Knill theorem assures that any quantum circuit composed exclusively from Clifford gates in $\mathcal{P}_2$, with computational basis preparation and measurement, may be efficiently simulated by a classical computer [7]. In contrast, incorporating any additional non-Clifford gate to $\mathcal{P}_2$ results in a universal gate set. In theory, gates in the Clifford group can be implemented with arbitrarily high precision by using concatenated stabilizer codes [8] or topological codes. Realistic systems also offer decoherence-free implementation of some Clifford gates. For this reason, it is important to fault-tolerantly perform *non-Clifford* logical gates outside of $\mathcal{P}_2$.

Summary of results

Let us now summarize the main contributions of this work. We provide a self-contained and arguably simpler derivation of BK's result. We derive a new technical lemma which is the key to assess fault-tolerant implementability of logical gates for both stabilizer and subsystem [20, 21] error-correcting codes. In addition, there are four original contributions which we now outline.

No-go result for self-correction.- First of all, we show that the property of self-correction imposes a further restriction on logical gates implementable by constant depth local circuits. Namely, we find that the assumption of having no string-like logical operators reduces the level of the implementable Clifford hierarchy by one with respect to BK's result.

Theorem 2. [Self-correction] *If a stabilizer Hamiltonian, consisting of geometrically-local bounded-norm terms in D spacial dimensions, has a macroscopic energy barrier, the set of logical gates, admitting a locality-preserving implementation, is restricted to $\mathcal{P}_{D-1}$.*

This theorem allows us to obtain a new no-go result for self-correcting quantum memory in three spatial dimensions; a three-dimensional topological stabilizer Hamiltonian with a locality-preserving non-Clifford gate cannot simultaneously have a macroscopic energy barrier. The result establishes a somewhat surprising connection between ground state properties and excitation energy landscape. While technically simple, this observation is arguably the most interesting.

Upper bound on code distance.- Our second result concerns a tradeoff between the code distance and locality-preserving implementability of logical gates. Namely, we find that implementability of logical gates from the higher-level Clifford hierarchy reduces an upper bound on the code distance of a topological stabilizer code.

Theorem 3. [Code distance] *If a stabilizer code with geometrically-local generators in D spatial dimensions admits a locality-preserving implementation of a logical gate $U \in \mathcal{P}_m$ for $m \geq 2$ (but $U \notin \mathcal{P}_{m-1}$), then its code distance is upper bounded by $d \leq O(L^{D+1-m})$.*

For a code with a non-Clifford gate ($m > 2$), this result improves the previous best bound $d \leq O(L^{D-1})$ for topological stabilizer codes [10]. The bound is found to be tight for $m = D$ as Bombin and Martin-Delgado's topological color codes saturates it [11, 12, 13, 14]. The theorem also applies to a topological subsystem code if its stabilizer subgroup admits a complete set of geometrically local generators. Such subsystem codes include Bombin's topological *gauge* color code [14].

Loss threshold.- Our third result relates the loss threshold in stabilizer and subsystem error-correcting codes with the set of transversally implementable logical gates.

Theorem 4. [Loss threshold] *Given a family of subsystem codes with a loss tolerance $p_l > 1/n$ for some natural number n. Then, any transversally implementable logical gate must belong to $\mathcal{P}_{n-1}$.*

We would like to emphasize that the above theorem does *not* assume geometric locality of generators or lattice structures, and holds for arbitrary stabilizer *and* subsystem codes.

Subsystem code and the Clifford hierarchy.- Finally, the main technical result is to generalize BK's result to subsystem codes with local generators. A difficulty is that the so-called union lemma does not apply to a topological subsystem code [15, 16]. Minimal supplementary assumptions, such as a finite loss threshold for the code and a logarithmically increasing code distance, are required in order to recover the same thesis as BK's for locality-preserving logical gates.

Theorem 5. [Subsystem code] *Consider a family of subsystem codes with geometrically local gauge generators in D spatial dimensions such that the code has a constant loss threshold and a code distance growing at least logarithmically in the number of physical qubits. Then, any locality-preserving logical unitary, fully supported on an m-dimensional region ($m \leq D$), has a logical action included in $\mathcal{P}_m$.*

Supplementary assumptions arise from considerations on fault-tolerance of the code. A finite loss threshold is necessary for a finite error threshold against depolarization. A logarithmically increasing code distance is necessary for the recovery failure probability to vanish at least polynomially in the number of physical qubits. Supplementary assumptions are not required for subsystem codes with geometrically local stabilizer generators as the union lemma holds for such codes.

Conclusions

We have provided several extensions of BK's characterization of fault-tolerantly implementable logical gates. Our results are summarized as follows: (i) A three-dimensional stabilizer Hamiltonian with a fault-tolerantly implementable non-Clifford gate is not self-correcting. (ii) The code distance of a D-dimensional topological stabilizer code with non-trivial m-th level logical gate is upper bounded by $O(L^{D+1-m})$. (iii) A loss threshold of a subsystem code with non-trivial m-th level transversal logical gate is upper bounded by $1/m$. (iv) Fault-tolerantly implementable logical gates in a D-dimensional topological subsystem code belong to the D-th level $\mathcal{P}_D$ in the presence of a finite error threshold.

While our results impose important constraints on the resources necessary to achieve universal fault-tolerant quantum computation they definitely do not exclude it. They are a guide to constructing a checklist of necessary resources. In particular, the non-local classical processing associated to gauge-fixing [22] in gauge-color codes [23] or for magic state distillation [24] or using complementary notions of transversality in concatenated codes [25] are some of the possible avenues to avoid the hypotheses of our results and achieve universal computation assuming reasonable sets of resources.

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