Universal operations in resource theories and local quantum thermodynamics

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Resource theories mathematically capture operational restrictions that naturally arise in various physical scenarios [1]. Two of the most paradigmatic examples of known resource theories are phenomenological thermodynamics – which roughly speaking reflects the resource theory of energy-conserving processes – and the theory of entanglement in quantum information theory. At the heart of the latter resource theory are those restrictions unavoidably arising in spatially separated multi-partite settings, allowing experimenters to only perform local operations with classical communication (LOCC) [2]. Another recently proposed resource theory in the quantum context, the resource theory of asymmetry, restricts experimenters to merely implement operations that are symmetric with respect to some group representation, giving rise to a framework of symmetry in quantum theory beyond the one captured in Noether's theorem [3].
Naturally, experimenters usually aim at performing certain specific tasks. The ultimate theoretical optimum degree to which this is possible when considering all operations that are allowed for is also provided by the resource theory. For example, they might want to extract work when running a thermal machine in the thermodynamic context, or they may wish to distill maximally entangled states within the LOCC framework. Given certain states as a provided resource, the resource theory delivers the maximal amount of work that can possibly be extracted or the maximum rate at which maximally entangled states can be extracted.
In our work we study how different resource theories interact. We consider proper subsets of all operations which are closed under composition (and contain the identity) and can hence be considered sub-theories [4]. If the sub-theory can still achieve the optimum for a specified task, we call it *universal* for the task. We then study whether this universality is stable against additional constraints that are being put on the operations.
As a case study we consider what is commonly called quantum thermodynamics as a resource theory and work extraction as a task, which we quantify in expectation value. Specifically, we consider situations in which an experimenter is allowed to perform an arbitrary sequence of the two following kind of operations: a) *Time evolution:* apply time-dependent Hamiltonian dynamics (which costs an amount of work given by average energy difference) and b) *Thermalising maps:* put the system into contact with a heat bath at inverse temperature β (which does not cost work). We consider three widely studied different models for the thermalising maps. The most general model is given by arbitrary Gibbs-preserving (GP) maps. A completely positive trace-

preserving (CPTP) map is called Gibbs-preserving with respect to a Hamiltonian H if it has the Gibbs-state $\omega_\beta(H) = Z_\beta(H)^{-1}\exp(-\beta H)$ as a fixed-point. Such maps contain physically motivated operations such as Markovian time-evolution with Gibbs-state as steady-state. They thus contain natural physical operations like putting the working system next to a heat bath for some time and removing it before it has fully thermalised. Depending on the microscopic details of the heat bath and the time-duration, this yields different outcomes on the same initial state. A proper subclass of GP maps is given by the so-called thermal operations (TO), as defined in [6, 7, 11]. Lastly, we consider the model that has been prevalent in the study of thermodynamics, give by the map, $(\rho, H) \mapsto (\omega_\beta(H), H)$, which we will refer to as *weak thermal contact* (WTC). Without any additional constraints, weak thermal contact is indeed universal for work extraction when combined with arbitrary time evolutions: Given any pair of an initial quantum state and a Hamiltonian (ρ, H) of the work system, the maximum amount of work $\langle W\rangle_{\text{opt}}^{\text{WTC}}(\rho, H)$ that can be extracted is given by the difference of non-equilibrium free energies compared to the Gibbs-state $\omega_\beta(H)$ [5, 8]:

$$\langle W\rangle_{\text{opt}}^{\text{WTC}}(\rho, H) = F_\beta(\rho, H) - F(\omega_\beta(H), H), \tag{1}$$

where $\beta^{-1} = k_B T$ and the non-equilibrium free energy is defined as $F_\beta(\rho, H) := \mathrm{Tr}(\rho H) - k_B T S(\rho)$ with $S(\rho)$ the von-Neumann entropy of ρ. The maximum work $\langle W\rangle_{\text{opt}}^{\text{WTC}}(\rho, H)$ that can be extracted using time evolutions and general Gibbs-reserving maps is also bounded from above by the free energy difference [9]. This is due to the fact that Gibbs-preserving operations can only decrease the relative-entropy distance to the Gibbs-state. Hence we have $\langle W\rangle_{\text{opt}}^{\text{WTC}}(\rho, H) = \langle W\rangle_{\text{opt}}^{\text{GP}}(\rho, H)$.

If we put additional constraints on the time evolutions the situation changes dramatically. The constraints are introduced as limitations on the set of accessible Hamiltonians. In particular, we consider i) a bound on the operator norm or ii) allowing the experimenter only to change local terms in an *interacting* Hamiltonian. The latter constraint defines what we call *local quantum thermodynamics*. Such an interplay between locality and thermodynamics has already yielded interesting results in restricted scenarios where constituents of the composite system are assumed to be non-interacting or work extraction is performed without access to heat baths [10]. Here, we address the problem in generality, investigating the effect of interacting Hamiltonians and thermal baths and formulating the problem from the viewpoint of resource theories.

In both of our constrained settings, the unitary time-evolutions that can be implemented on the full system are the same as in the unconstrained setting. For case i) this is clear since scaling a Hamiltonian is the same as changing the time-scale. For the situation where we have a constrain on locality this is less obvious. Using ideas from quantum control theory, we prove that *any* interaction together with the ability to change local Hamiltonians can be used to effectively implement arbitrary global unitaries in the infinite-time limit. Results from control theory suggest that such results hold for many constraints that one can put on the allowed Hamiltonians [12].

Yet, surprisingly, despite the fact that the unitary operations that the experimenters can apply on the system are unchanged in both settings, weak thermal contact completely looses its universality: There exist initial pairs $(\tilde{\rho}, \tilde{H})$ from which no work can be extracted with weak thermal contact but some amount of work can be extracted using more general Gibbs-preserving maps:

$$\langle \tilde{W} \rangle_{\text{opt}}^{\text{WTC}}(\tilde{\rho}, \tilde{H}) \leq 0 < \langle \tilde{W} \rangle_{\text{opt}}^{\text{GP}}(\tilde{\rho}, \tilde{H}), \tag{2}$$

here $\langle \tilde{W} \rangle_{\text{opt}}$ denotes the maximum amount of work that can be extracted in the respective constrained scenario.

Similarly to Ref. [6], to establish our results we found it necessary to enlarge the state-space from quantum states to pairs of quantum states and Hamiltonians. This is due to the interplay between the set of allowed Hamiltonians and the Gibbs-preserving maps: Since the property of being Gibbs-preserving is defined relative to a Hamiltonian, the set of operations that are allowed at a certain moment of time depend on the current Hamiltonian of the working-system. If we constrain the set of Hamiltonians that the experimenter can reach, the set of Gibbs-states and Gibbs-preserving maps that can be reached during a work-extraction protocol is also constrained. Since the allowed unitary operations are unchanged, it is precisely this interplay that is the only reason for the smaller amount of work that can be extracted in the constrained settings.

The example for the gap in work-extraction in the situation with the locality-constraint relies on the existence of special interacting Hamiltonians whose equilibrium free energy can only be decreased by adding local terms. This is sufficient to show that there exist intial configurations $(\tilde{\rho}, \tilde{H})$ from which no work can be extracted using WTC and time evolutions with Hamiltonians restricted as defined in ii). Such Hamiltonians also exist for other constraints and we prove that a sufficient condition for this property is that the Gibbs-state of the Hamiltonian is Hilbert-Schmidt orthogonal to all other Hamiltonians (up to their traces) the experimenter can go to from the initial Hamiltonian.

Our work can be seen as a study on how operational constraints (and hence resource theories) interact in a setting of practical physical interest: locality and thermodynamics. It has already been shown before that quantum thermodynamics (although in a different formulation) can be understood as an intersection between the resource theories of purity and asymmetry [15]. We hope that further research in this direction will elucidate the formal and practical connections between different resource theories.

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